

# APERIODIC SEQUENCES AND APERIODIC GEODESICS

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**ABSTRACT.** We introduce a quantitative condition on orbits of dynamical systems which measures their aperiodicity. We show the existence of sequences in the Bernoulli-shift and geodesics on closed hyperbolic manifolds which are as aperiodic as possible with respect to this condition.

## 1. MAIN RESULTS.

In this section we state our main results in the case of sequences in a finite alphabet and of geodesics in hyperbolic manifolds. Denote by  $\mathbb{N}_0$  the natural numbers including 0 and let  $\mathbb{N} = \mathbb{N} \setminus \{0\}$ . Given a finite set  $\mathcal{A}$  with  $k \geq 2$  elements, let  $\Sigma = \mathcal{A}^{\mathbb{Z}}$  be the set of biinfinite sequences in the alphabet  $\mathcal{A}$ , which we call *words*. With  $[w(i) \dots w(i+l)]$  denote the subword of  $w \in \Sigma$  starting at *time*  $i \in \mathbb{Z}$  and of *length*  $l \in \mathbb{N}_0$ . For a word  $w \in \Sigma$  define the *recurrence time*  $R_w^i : \mathbb{N}_0 \rightarrow \mathbb{N} \cup \{\infty\}$  at time  $i \in \mathbb{Z}$  by

$$R_w^i(l) = \min\{s \geq 1 : [w(i+s) \dots w(i+s+l)] = [w(i) \dots w(i+l)]\},$$

(i.e. the first instant when the sub word  $[w(i) \dots w(i+l)]$  of  $w$  is seen again), and by

$$R_w(l) := \min\{R_w^i(l) : i \in \mathbb{Z}\}.$$

For a periodic word  $w \in \Sigma$  with period  $p \in \mathbb{N}$ , i.e.  $w(i) = w(i+p)$  for all  $i \in \mathbb{Z}$ , we have  $R_w(l) \leq p$  for all  $l \in \mathbb{N}_0$ . Thus, if  $R_w$  is unbounded, then  $w$  is aperiodic and we view the growth rate of  $R_w$  as a measure for the aperiodicity of the word  $w$ . Note that  $R_w$  is nondecreasing and by a trivial counting argument we have  $R_w(l) \leq k^{l+1}$  for every word  $w$ , in particular

$$\lim_{l \rightarrow \infty} \frac{1}{l} \ln R_w(l) \leq \ln(k).$$

One of our main results is the existence of words  $w$  such that the growth rate is as near as possible to this bound.

**Theorem 1.1.** *Let  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing function such that*

$$\lim_{l \rightarrow \infty} \frac{1}{l} \ln(\varphi(l)) \leq \delta \ln(k) \tag{1.1}$$

*for some  $0 < \delta < 1$ . Then there exist  $l_0 = l_0(\varphi, k, \delta) \in \mathbb{N}_0$  and a word  $w \in \Sigma$  such that, for every  $l_0 \leq l \in \mathbb{N}_0$ , we have  $R_w(l) \geq \varphi(l)$ .*

Now let  $M$  be a closed  $n$ -dimensional hyperbolic manifold, where  $n \geq 2$ . Let  $i_M > 0$  denote the injectivity radius of  $M$  and let  $d$  be the Riemannian distance function on  $M$ . For

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a unit speed geodesic  $\gamma : \mathbb{R} \rightarrow M$  we define the *recurrence time*  $R_\gamma^{t_0} : [0, \infty) \rightarrow [i_M/2, \infty]$  at time  $t_0 \in \mathbb{R}$  by

$$R_\gamma^{t_0}(l) = \inf\{s > i_M/2 : d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \frac{i_M}{2} \text{ for all } 0 \leq t \leq l\}.$$

and

$$R_\gamma(l) := \inf\{R_\gamma^{t_0}(l) : t_0 \in \mathbb{R}\}.$$

If  $\gamma$  is a periodic geodesic, then  $R_\gamma$  is bounded and again one can view the growth rate of  $R_\gamma$  as a measure for the aperiodicity of  $\gamma$ .

**Theorem 1.2.** *Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a non-decreasing function such that*

$$\lim_{l \rightarrow \infty} \frac{1}{l} \ln(\varphi(l)) \leq \delta(n-1) \quad (1.2)$$

*for some  $0 < \delta < 1$ . If  $i_M > 2 \ln(2)$  then there exist  $l_0 = l_0(\varphi, \delta, n, i_M) \geq 0$  and a unit speed geodesic  $\gamma : \mathbb{R} \rightarrow M$  such that for all  $l \geq l_0$ , we have  $R_\gamma(l) \geq \varphi(l)$ .*

The theorems will be shown in greater generality.

*Remark.* The bounds  $\ln(k)$  and  $n-1$  equal the topological entropies of the respective dynamical systems. Moreover, we believe that the assumption on the injectivity radius in Theorem 1.2 is not necessary. A version of this theorem is also true if  $M$  is of strictly negative curvature. However, for the sake of clarity of the paper we restrict to these assumptions.

*Organization of the paper.* In Section 2 we will introduce the measure of aperiodicity for general dynamical systems and deduce immediate properties. In Section 3 and 4 we examine two examples and state the main results, namely of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold. These will be proven in Section 5.

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## 2. $F$ -APERIODIC POINTS.

Let  $(X, d)$  be a compact metric space and let  $T : X \rightarrow X$  be a given continuous transformation. For  $n \in \mathbb{N}_0$  let  $T^n$  be the  $n$ -times composition of  $T$  (where  $T^0 = id_X$ ) and for a point  $x \in X$  let  $T^n x$  be the point in the orbit  $\mathcal{T}(x) := \{T^n x\}_{n \in \mathbb{N}_0}$  of  $x$  at time  $n$ . Let moreover  $\mu$  be a finite Borel-measure on the Borel- $\sigma$ -algebra  $\mathcal{B}$  of  $(X, d)$  such that  $T$  is measure-preserving; see [5].

A point  $x \in X$  is called *periodic* (with respect to  $T$ ) if there exists an integer  $p \in \mathbb{N}$ , called a *period* of  $x$ , such that  $T^p x = x$ . Denote by  $\mathcal{P}_T$  the  $T$ -invariant set of  $T$ -periodic points of  $X$ . A point is called *aperiodic*, if it is not periodic.

A point  $x \in X$  is *recurrent* with respect to  $T$ , if for any  $\varepsilon > 0$  there exists  $s = s(x, \varepsilon) \in \mathbb{N}$  such that  $d(T^s x, x) < \varepsilon$ . Periodic points are obviously recurrent. The set  $\mathcal{R}_T$  of recurrent points is nonempty (see [6]) and  $T$ -invariant. However  $s(T^i x, \varepsilon)$  can differ from  $s(x, \varepsilon)$  in general, unless  $T$  is an isometry on its orbit  $\mathcal{T}(x)$ ; that is,  $d(T^{i+s} x, T^i x) = d(T^s x, x)$  for all  $i$  and  $s \in \mathbb{N}_0$ . We recall that by the Poincaré-recurrence theorem,  $\mu$ -almost every point is recurrent.

In this paper we give a quantitative version of recurrence and aperiodicity. Given a point  $x \in X$  and a time  $i \in \mathbb{N}_0$ , we ask for a lower bound on the *shift*  $s$  such that  $T^{i+s}x$  is allowed to be  $\varepsilon$ -close to  $T^i x$ :

**Definition 2.1.** For a non-increasing function  $F : (0, \infty) \rightarrow [0, \infty)$  a point  $x \in X$  is called *F-aperiodic* at time  $i \in \mathbb{N}_0$  if for every  $\varepsilon > 0$ , whenever

$$d(T^i x, T^{i+s} x) < \varepsilon$$

for some  $s \in \mathbb{N}$ , then  $s > F(\varepsilon)$ . If  $x$  is *F-aperiodic* at every time  $i \in \mathbb{N}_0$  then it is called *F-aperiodic*.

We emphasize that although we called the condition "F-aperiodic", a periodic point  $x$  is *F-aperiodic* for a suitable bounded function  $F$ . However, if the function  $F$  is unbounded, an *F-aperiodic* point must be aperiodic. Moreover, if  $x$  is not recurrent, then it is easy to find an unbounded function  $F$  such that  $x$  is *F-aperiodic* at least at time 0.

Let  $F : (0, \infty) \rightarrow [0, \infty)$  be a given non-increasing function. Clearly, if a non-increasing function  $\bar{F}$  satisfies  $\bar{F}(s) \leq F(s)$  for all  $s \in (0, \infty)$  then an *F-aperiodic* point is also  $\bar{F}$ -aperiodic. On the other hand, using the upper box dimension  $\dim_B(X)$  for metric spaces, we obtain an upper bound on the growth rate (as  $\varepsilon$  tends to 0) of functions  $F$  such that an *F-aperiodic* point might exist. For  $\varepsilon > 0$  let  $N(X, \varepsilon)$  denote the largest number of disjoint metric balls of radius  $\varepsilon$ . Then the upper box dimension ([16]) is given by

$$\dim_B(X) = \limsup_{\varepsilon \rightarrow 0} \frac{\ln(N(X, \varepsilon))}{-\ln(\varepsilon)}.$$

**Lemma 2.2.** *Let  $x$  be an F-aperiodic point. Then  $\limsup_{\varepsilon \rightarrow 0} \frac{\ln(F(\varepsilon))}{\ln(2/\varepsilon)} \leq \dim_B(X)$ .*

*Proof.* Let  $\varepsilon > 0$ . If  $B(T^{s_1}x, \varepsilon/2) \cap B(T^{s_2}x, \varepsilon/2) \neq \emptyset$  for some  $0 \leq s_1 < s_2 \leq F(\varepsilon)$ , we have  $d(T^{s_1}x, T^{s_2}x) < \varepsilon_0$  which is impossible since  $s_2 - s_1 \leq F(\varepsilon_0)$ . Therefore the metric balls  $B(T^s x, \varepsilon/2)$  must be disjoint for  $s \leq F(\varepsilon)$ . Hence we have  $F(\varepsilon) \leq N(X, \varepsilon/2)$ .  $\square$

Moreover, since  $F$  is independent of the time  $i \in \mathbb{N}_0$ , the set  $\mathcal{F}_T \subset X$  of *F-aperiodic* points is  $T$ -invariant. In the case when  $(X, \mathcal{B}, \mu, T)$  is ergodic,  $\mathcal{F}_T$  is either of full or of zero  $\mu$ -measure. When  $\mathcal{P}_T$  is nonempty, this question is related to the distribution of periodic orbits. In fact, let  $x_0 \in \mathcal{P}_T$  be of minimal period  $p_0$  and assume that  $F(\varepsilon) \geq p_0$  for some  $\varepsilon_{p_0} > 0$ . In the case when  $F$  is continuous, we may choose  $\varepsilon_{p_0} := \sup\{\varepsilon > 0 : F(\varepsilon) \geq p_0\}$ . Define the *critical neighborhood* of  $x_0$  with respect to  $F$  and  $p_0$  by

$$\mathcal{N}_{x_0} := B(x_0, \varepsilon_{p_0}/2) \cap T^{-p_0}(B(x_0, \varepsilon_{p_0}/2)). \quad (2.1)$$

Whenever  $x \in \mathcal{N}_{x_0}$  we have by the triangle inequality that  $d(x, T^{p_0}x) < \varepsilon_{p_0}$ , but  $p_0 \leq F(\varepsilon_{p_0})$ . Thus, no point in  $\mathcal{N}_{x_0}$  can be *F-aperiodic* and we see that the orbit of an *F-aperiodic* point must avoid the critical neighborhoods of periodic points. If in addition  $\mu(\mathcal{N}_{x_0}) > 0$  then the set of *F-aperiodic* points cannot be of full and must therefore be of zero  $\mu$ -measure. Thus, we showed the following criterion.

**Lemma 2.3.** *Assume  $\mathcal{P}_T \neq \emptyset$  and let  $x_0$  be a periodic point of period  $p_0$  and  $F(\varepsilon) \geq p_0$  for some  $\varepsilon > 0$ . If  $\mu$  is ergodic and positive on  $\mathcal{N}_{x_0}$  then the set  $\mathcal{F}_T$  has  $\mu$ -measure 0.*

In particular, this result is interesting for the *systolic point*  $x_0 \in \mathcal{P}_T$  of *systolic period*  $p_0 \in \mathbb{N}$ , that is,  $x_0$  has minimal period  $p_0$  and for every periodic point in  $X$  of period  $p$  we have  $p \geq p_0$ .

**Lemma 2.4.** *F-aperiodicity is a closed condition.*

*Proof.* Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of  $F$ -aperiodic points in  $X$  converging to  $x \in X$ . Let  $i$  and  $s \in \mathbb{N}$  be fixed. For  $\varepsilon > 0$  such that  $d(T^i x, T^{i+s} x) < \varepsilon$  let  $d := \frac{1}{2}(\varepsilon - d(T^i x, T^{i+s} x))$ . Since  $T$  is continuous, there exists  $N = N(i, s, d) \in \mathbb{N}_0$  such that for all  $n \geq N$  we have  $d(T^i x, T^i x_n) < d$  and  $d(T^{i+s} x, T^{i+s} x_n) < d$ . From the triangle inequality we obtain

$$d(T^i x_n, T^{i+s} x_n) \leq d(T^i x_n, T^i x) + d(T^i x, T^{i+s} x) + d(T^{i+s} x, T^{i+s} x_n) < \varepsilon$$

for  $n \geq N$  so that  $s > F(\varepsilon)$  since  $x_n$  is  $F$ -aperiodic. Hence,  $x$  is also  $F$ -aperiodic.  $\square$

Finally, note that if  $T$  acts as an isometry on the orbit  $\mathcal{T}(x)$  of a point  $x \in X$ , then  $x$  is  $F$ -aperiodic as soon as it is  $F$ -aperiodic at a given time. For instance, we consider the rotation on the circle as a motivating example:

**Example 1.** Let  $\mathbb{Z}$  act on  $\mathbb{R}$  by translations and let  $X = \mathbb{R}/\mathbb{Z}$  be the compact quotient space with the induced metric  $d$  obtained from the Euclidean metric. Given an irrational number  $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$ , we let  $T = T_\alpha : X \rightarrow X$  be the automorphism induced by the translation  $\tilde{T} : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\tilde{T}(x) := x + \alpha$ . For  $c > 0$  we let  $F_c : (0, \infty) \rightarrow [0, \infty)$ ,  $F_c(t) = ct^{-1}$ . In fact, since  $\dim_B(X) = 1$ ,  $-1$  is the optimal exponent due to Lemma 2.2. The point  $[0]$  is  $F_c$ -aperiodic if and only if every point  $[x]$  is  $F_c$ -aperiodic and hence  $\mathcal{F}_T$  is either empty or  $X$  itself. Moreover, since  $T$  is an isometry,  $[0]$  is  $F_c$ -aperiodic as soon as it is  $F_c$ -aperiodic at time 0. The question for which  $c$  and  $\alpha$  there exist  $F_c$ -aperiodic points can be answered by classical Diophantine approximation; see for instance [1] for the following well-known results: Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$ . For  $\mu$ -almost every  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  we have  $c_0(\alpha) = 0$ , where

$$c_0(\alpha) = \inf\{c > 0 : \text{there exist infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } |\alpha - \frac{p}{q}| < \frac{c}{q^2}\}.$$

However, there exists a set of Hausdorff-dimension one such that  $c_0(\alpha)$  is positive. Such an  $\alpha$  is called badly approximable. The supremum  $\sup_{\alpha \in \mathbb{R} \setminus \mathbb{Q}} c_0(\alpha)$  of this set, called the Hurwitz-constant, is equal to  $1/\sqrt{5}$  and attained at the golden ratio.

First, let  $\alpha$  such that  $c_0(\alpha) = 0$ . Then for  $c > 0$  we have for infinitely many  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ ,

$$|\tilde{T}^q 0 - p| = |q\alpha - p| = q|\alpha - \frac{p}{q}| < cq^{-1}, \quad (2.2)$$

hence  $q \leq F_c(cq^{-1})$  and we see that  $[0]$  is not  $F_c$ -aperiodic for any  $c > 0$ . Thus,  $\mathcal{F}_T$  is empty. In particular, this shows that for  $c > 1/\sqrt{5}$  the set  $\mathcal{F}_T$  is empty for every  $T = T_\alpha$ ,  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  irrational. However, for  $\alpha$  a badly approximable number we have  $c_0(\alpha) > 0$  and for  $c < c_0(\alpha)$  there are only finitely many  $p, q$  as in (2.2). Hence we can choose some  $0 < \bar{c} \leq c_0(\alpha)$  such that  $[0]$  is  $F_{\bar{c}}$ -aperiodic and therefore  $\mathcal{F}_T = X$ .

If we conversely assume that  $[0]$  is  $F_c$ -aperiodic, then whenever  $|\tilde{T}^q 0 - p| < \varepsilon$  for some  $\varepsilon > 0$  we have  $q > F_c(\varepsilon) = c/\varepsilon > \frac{c}{q|\alpha - p/q|}$ . Thus,  $|\alpha - \frac{p}{q}| > \frac{c}{q^2}$  for every  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$  and  $\alpha$  is necessarily a badly approximable number.

In the following we are concerned with the examples of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold where the question of existence of  $F$ -aperiodic points is more delicate.

*Remark.* A somewhat orthogonal problem has been studied by many authors. For instance, [2] showed that the rate of recurrence can be quantified in the case when  $X$  has finite Hausdorff-dimension. More precisely, assume that the  $\alpha$ -dimensional Hausdorff-measure

$H_\alpha$  is  $\sigma$ -finite for some  $\alpha > 0$ , then for  $\mu$ -almost every point  $x \in X$  there exists a finite constant  $c(x) \geq 0$  such that

$$\liminf_{n \rightarrow \infty} n^{1/\alpha} d(x, T^n(x)) \leq c(x).$$

Assume that there exists a point  $x \in X$  which is  $F$ -aperiodic at time 0 for the function  $F(\varepsilon) = c \cdot \varepsilon^{-\alpha}$  for some  $c > 0$  (compare with Lemma 2.2). Then it is not hard to show that for every  $n > 0$ ,

$$n^{1/\alpha} d(x, T^n x) \geq c^{1/\alpha}.$$

The main point in our paper is that we study the recurrence for every point of the orbit and not only for the initial one.

### 3. SEQUENCES.

Let  $\mathcal{A}$  be a finite set of  $k \geq 2$  elements which we call *alphabet*. Let  $\Sigma^+ = \{w : \mathbb{N} \rightarrow \mathcal{A}\}$  and  $\Sigma = \{w : \mathbb{Z} \rightarrow \mathcal{A}\}$  be the set two-sided sequences in symbols from  $\mathcal{A}$ . The elements of  $\Sigma$  are called *words*. Given words  $w$  and  $\bar{w}$  in  $\Sigma$  we let  $a(w, \bar{w}) = \max\{i \geq 0 : w(i) = \bar{w}(i) \text{ for } |j| \leq i\}$  for  $w \neq \bar{w}$  and define  $\bar{d}(w, \bar{w}) := 2^{-a(w, \bar{w})}$ , and  $\bar{d}(w, w) := 0$  otherwise. Let  $T$  denote the shift operator acting on  $\Sigma$ , with  $T(w) = \bar{w}$  where  $\bar{w}(i) = w(i+1)$ . Then,  $(\Sigma, \bar{d})$  is a compact metric space such that  $T$  is a homeomorphism. Moreover, let  $\mathcal{B}$  denote the product  $\sigma$ -algebra of the power set  $\mathcal{P}(\mathcal{A})$  of  $\mathcal{A}$  which equals the Borel- $\sigma$ -algebra of  $(\Sigma, \bar{d})$ . Let (the probability measure)  $\mu = \prod_{\mathbb{Z}} \mu_{\mathcal{A}}$  be the infinite product measure of  $\mathcal{B}$  where  $\mu_{\mathcal{A}}$  is a probability measure on  $(\mathcal{A}, \mathcal{P}(\mathcal{A}))$ . Then the *Bernoulli-shift*  $(\Sigma, \mathcal{B}, \mu, T)$  is ergodic. For details we refer to [5].

Note that by definition of  $\bar{d}$ , two words are close if and only if the length of their subwords around position 0 on which they agree is large. In particular, if  $w \in \mathcal{R}_T$  then, by recurrence applied to the word  $T^i w$ , for every length  $l \in \mathbb{N}_0$  we can find an  $s = s(i, l) \in \mathbb{N}$  such that  $[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$ . In the case of sequences it is suitable to reformulate  $F$ -aperiodicity as follows (see Proposition 3.2).

**Definition 3.1.** For a non-decreasing function  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$  a word  $w \in \Sigma$  is called  $\varphi$ -aperiodic at time  $i \in \mathbb{Z}$ , if for every length  $l \in \mathbb{N}_0$ , whenever

$$[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)] \quad (3.1)$$

for some *shift*  $s \in \mathbb{N}$ , then  $s > \varphi(l)$ . If  $w$  is  $\varphi$ -aperiodic at every time  $i \in \mathbb{Z}$  it is called  $\varphi$ -aperiodic.

A  $\varphi$ -aperiodic word  $w \in \Sigma$  is  $F$ -aperiodic for the following function  $F$ .

**Proposition 3.2.** A  $\varphi$ -aperiodic word  $w \in \Sigma$  is  $F$ -aperiodic for  $F(\varepsilon) = \varphi(-2\lceil \log_2(\varepsilon) \rceil)$ . Conversely, an  $F$ -aperiodic word  $w$  is  $\varphi$ -aperiodic for  $\varphi(l) = F(2^{-(l/2-1)})$ .

*Proof.* Let  $i \in \mathbb{Z}$  and  $s \in \mathbb{N}$ . For every  $l \in \mathbb{N}_0$  such that  $\bar{d}(T^i w, T^{i+s} w) \leq 2^{-l}$  we have  $[w(i-l) \dots w(i+l)] = [w(i-l+s) \dots w(i+s+l)]$ . Thus, for  $2^{-l} < \varepsilon \leq 2^{-(l-1)}$ ,

$$s > \varphi(2l) = \varphi(-2\lceil \log_2(\varepsilon) \rceil) = F(\varepsilon).$$

Since  $F(\bar{\varepsilon}) \leq F(\varepsilon)$  for  $\bar{\varepsilon} \geq \varepsilon$ , the first implication follows.

Conversely, if  $w$  is  $F$ -aperiodic, assume that  $[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$  for  $s \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$  and let  $\bar{l} = l/2$  if  $l$  is even and  $\bar{l} = (l-1)/2$  if  $l$  is odd. Hence,  $\bar{d}(T^{i+\bar{l}} w, T^{i+\bar{l}+s} w) \leq 2^{-\bar{l}}$  and for every  $2^{-\bar{l}} < \varepsilon \leq 2^{-(\bar{l}-1)}$  we have

$$s > F(\varepsilon) \geq F(2^{-(\bar{l}-1)}) \geq F(2^{-(l-3)/2}) = \varphi(l).$$

This finishes the proof.  $\square$

If a  $\varphi$ -aperiodic word contains a periodic subword of infinite length then the function  $\varphi$  is bounded, whereas if a word is  $\varphi$ -aperiodic for an unbounded function, the word must be aperiodic. We want to give some examples in order to make the definition more familiar, among them the prominent Morse-Thue-sequence:

**Example 2.** First, let  $a, b \in \mathcal{A}$ . One checks that the (non-recurrent) words  $w_1 = \dots bbbaaa \dots$  and  $w_2 = \dots abaabaaabaaaab \dots$  are  $\varphi$ -aperiodic only for a function  $\varphi$  such that  $1 = s > \varphi(l)$  for all  $l \in \mathbb{N}_0$ . Both, the orbits of  $w_1$  and  $w_2$ , come closer and closer to the periodic word  $\dots aaa \dots$  with respect to the metric  $\bar{d}$ . This is not the case for  $\varphi$ -aperiodic words when  $\varphi$  is unbounded; see Proposition 3.4.

Consider the Morse-Thue recurrent sequence  $w \in \{0, 1\}^{\mathbb{Z}}$  which is determined as follows: Let  $a_0 = 0, b_0 = 1$ . Then for  $n \in \mathbb{N}_0$ , let  $a_{n+1} = a_n b_n$  and  $b_{n+1} = b_n a_n$  be finite words of length  $2^{n+1} - 1$ . Then  $w$  is defined such that it satisfies  $[w(0) \dots w(2^n - 2)] = a_n$  and  $[w(-n)] = [w(n - 1)]$  for every  $n \in \mathbb{N}$ . In particular,  $w$  contains the sub words  $a_{n+2} = a_n b_n b_n a_n$ . Hence for every length  $l = 2^n - 1$ ,  $w$  contains subwords of the form  $WW$  where  $W$  has length  $l$ . A function  $\varphi$  such that  $w$  is  $\varphi$ -aperiodic must therefore be bounded by  $\varphi(2^n - 1) \leq 2^n - 1$  for every  $n \in \mathbb{N}$ . On the other hand there are no sub words of the form  $WWa$  where  $a$  is the first letter of a sub word  $W$  (see [11]). In other words,  $w$  is overlap-free (which means that there are no sub words of the form  $aWaWa$  for a finite sub word  $W$  and a letter  $a$ ), from which follows that there are even no sub words of the form  $wWwWw$  for  $w$  and  $W$  finite subwords. Hence we may choose  $\varphi(l) \geq l$ . We conclude that  $w$  is at least  $\varphi$ -aperiodic for the function  $\varphi(l) = l, l \in \mathbb{N}_0$ .

The example shows that the set of  $\varphi$ -aperiodic words  $\mathcal{F}_T = \mathcal{F}_T(\varphi)$  is nonempty for the unbounded function  $\varphi(l) = l$  and moreover, the Morse-Thue sequence gives an explicit example of such a word. However, let  $a \in \mathcal{A}$  such that  $\mu_{\mathcal{A}}(\{a\}) > 0$  and let  $w = \dots aaa \dots$  be a periodic word which is of systolic period 1. Moreover,  $\mu$  is positive on the critical neighborhood of  $w$  and hence by Lemma 2.3,  $\mathcal{F}_T$  is of zero  $\mu$ -measure unless  $\varphi$  is strictly bounded by 1.

Our main result for sequences is the following. It will be proved in Section 5.

**Theorem 3.3.** *Let  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing unbounded function such that there exists  $c \in (1, k)$  satisfying*

$$k - \lfloor \varphi(0) \rfloor - \sum_{l=1}^{\infty} \frac{\lfloor \varphi(l) \rfloor - \lfloor \varphi(l-1) \rfloor}{c^l} \geq c, \quad (3.2)$$

where  $\lfloor \cdot \rfloor$  denotes the integer part. Then there exists a  $\varphi$ -aperiodic word in  $\Sigma$ .

**Remark.** The condition is satisfied for the following set of parameters:

- (1)  $k \geq 4$ , then  $\varphi(l) = l$  satisfies (3.2) for  $c = 2$ ,
- (2)  $k \geq 5$ , then  $\varphi(l) = 2^l$  satisfies (3.2) for  $c = 3$ ,
- (3)  $k \geq 2, 0 < \delta < 1$  and  $k^\delta < c < k$ , then there exists  $l_0 = l_0(k, \delta, c) \in \mathbb{N}_0$  such that

$$\varphi(l) = \begin{cases} 0, & \text{for } l \leq l_0 \\ k^{\delta l}, & \text{for } l > l_0 \end{cases} \quad (3.3)$$

satisfies (3.2).

Note that if a word  $w$  is  $\varphi$ -aperiodic then  $R_w(l) > \varphi(l)$  for every  $l \in \mathbb{N}_0$  where  $R_w$  is the recurrence time introduced in Paragraph 1. Theorem 1.1 is hence a corollary of Theorem 3.3.

*Proof of Theorem 1.1.* By condition (1.1), for every  $\varepsilon_0 > 0$  there exists  $l_1 = l_1(\varepsilon_0) \in \mathbb{N}$  such that for all  $l \geq l_1$ ,

$$\frac{1}{l} \ln(\varphi(l)) \leq \delta \ln(k)(1 + \varepsilon_0).$$

Since  $\delta < 1$  we let  $\varepsilon_0 > 0$  such that  $\tilde{\delta} = (1 + \varepsilon_0)\delta < 1$ . Then,  $\varphi(l) \leq k^{\tilde{\delta}l}$  for  $l \geq l_1$ . If we take  $c := (k - k^{\tilde{\delta}})/2$  then by (3.3) there exists  $l_2 = l_2(k, \tilde{\delta})$  such that condition (3.2) is satisfied for the function  $\bar{\varphi}(l) := k^{\tilde{\delta}l}$  for  $l > l_2$  and  $\bar{\varphi}(l) = 0$  for  $l \leq l_2$ ,  $l \in \mathbb{N}_0$ . Theorem 3.3 implies the existence of a  $\bar{\varphi}$ -aperiodic word  $w \in \Sigma$ . Thus, setting  $l_0 := \max\{l_1, l_2\} + 1$ , we have that  $\bar{\varphi}(l) \geq \varphi(l)$  for all  $l \geq l_0$  and the claim follows.  $\square$

*Remark.* The critical function  $\varphi$  for which  $\varphi$ -aperiodic words cannot exist is the function  $\varphi(l) = k^{l+1}$ . The critical exponent  $\ln(k)$  equals the topological entropy of the system  $(\Sigma, \bar{d}, T)$  (see [20]) and is optimal. To see that there exists no  $w \in \Sigma$  which is  $\varphi$ -aperiodic for a function  $\varphi$  such that  $\varphi(l) \geq k^{l+1} - 1$  for some  $l \in \mathbb{N}_0$ , fix a subword  $[w(1) \dots w(1+l)]$  of any  $w \in \Sigma$ . Inductively one shows that at each step  $1 \leq s \leq \varphi(l)$  one has at most  $k^{l+1} - s$  possibilities to choose a sub word  $[w(1+s) \dots w(1+s+l)]$  such that  $w$  stays  $\varphi$ -aperiodic. Then, at step  $s = k^{l+1}$ , there is no choice left such that  $w$  is  $\varphi$ -aperiodic.

*Remark.* Let  $\Sigma^+(m) = \{w : \{1, \dots, m\} \rightarrow \mathcal{A}\}$  be the set of words of length  $m$  in  $\mathcal{A}$  and  $\mathcal{W}^g(m) \subset \Sigma^+(m)$  be the set of *good* words of length  $m$  which satisfy (3.1) for all  $i, s \in \mathbb{N}$  and  $l \in \mathbb{N}_0$  such that  $i + s + l \leq m$ . If  $\varphi$  satisfies (3.2) with respect to the parameter  $c > 1$  we will see in the proof of Theorem 3.3 (see Lemma 5.6) that the good words  $\mathcal{W}^g(m)$  increase in  $m$  by the factor  $c$ . Thus,  $|\mathcal{W}^g(m)| \geq c^m$  which is a lower bound on the asymptotic growth of  $|\mathcal{W}^g(m)|$ , where  $|\cdot|$  denotes its cardinality.

We may reformulate the critical neighborhood of a periodic point given in (2.1) to the setting of  $\varphi$ -aperiodicity. Moreover, since  $\mathcal{P}_T$  is dense in  $\Sigma$  we can also give a sufficient condition on  $\varphi$ -aperiodicity in terms of periodic words. Therefore, for a non-decreasing unbounded function  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$ , we define a discrete form of a right-inverse for  $\varphi$  by  $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ ,

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\}, \quad (3.4)$$

which is also non-decreasing and unbounded.

**Proposition 3.4.** *Let  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing unbounded function. If  $w \in \Sigma$  is  $\varphi$ -aperiodic, then for every periodic word  $\bar{w} \in \Sigma$  of period  $s$  and for all  $i \in \mathbb{Z}$  we have*

$$\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s))/2}.$$

*Conversely, if  $\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s)-1)/2}$  for every periodic word  $\bar{w}$  of period  $s$  and all  $i \in \mathbb{Z}$ , then  $w$  is  $\varphi$ -aperiodic.*

*Proof.* If  $w$  is  $\varphi$ -aperiodic,  $w$  is aperiodic and there exists  $m \in \mathbb{N}_0$  such that  $\bar{d}(T^i w, \bar{w}) = 2^{-m}$  where we assume  $2m \geq s$  (otherwise the first statement follows). Hence,  $[w(i-m) \dots w(i+m)] = [\bar{w}(-m) \dots \bar{w}(m)]$  and we see that  $[w(i-m) \dots w(i-m+s+(2m-s))] = [w(i-m+s) \dots w(i+m)]$ . Thus,  $s > \varphi(2m-s)$  and  $m < (s+\ell(s))/2$  from (5.1).

Conversely, assume that  $[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$  for  $s \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$  and let  $\bar{l} = (s+l)/2$  if  $s+l$  even and  $\bar{l} = (s+l-1)/2$  if  $s+l$  is odd. Moreover, let  $\bar{w}$  be the periodic word of period  $s$  such that  $[\bar{w}(i) \dots \bar{w}(i+s-1)] = [w(i) \dots w(i+s-1)]$ . Thus,  $2^{-\bar{l}} \geq d(T^{i+\bar{l}} w, T^{i+\bar{l}} \bar{w}) > 2^{-(s+\ell(s)-1)/2}$  and we see that  $s + \ell(s) - 1 > 2\bar{l} \geq s + l - 1$ . Hence,  $l < \ell(s)$  and from (5.1) we have  $s > \varphi(l)$ .  $\square$

*Remark.* Consider the overlap-free recurrence time  $\tilde{R}_w^0 : \mathbb{N}_0 \rightarrow \mathbb{N}$  of the initial sub word,

$$\tilde{R}_w^0(l) = \min\{s > l : [w(s) \dots w(s+l)] = [w(0) \dots w(l)]\}.$$

Clearly,  $R_w(l) \leq R_w^0(l) \leq \tilde{R}_w^0(l)$  for  $l \in \mathbb{N}_0$ . Then it follows from [12] that, since the Bernoulli-shift is ergodic, for  $\mu$ -almost all  $w \in \Sigma$  the limit

$$\lim_{l \rightarrow \infty} \frac{\ln \tilde{R}_w^0(l)}{l}$$

exists and equals the measure-entropy  $h_\mu(T)$ .

#### 4. GEODESIC FLOW ON HYPERBOLIC MANIFOLDS

Let  $M$  be a closed  $n$ -dimensional hyperbolic manifold, that is a compact connected Riemannian manifold without boundary of constant negative curvature  $-1$ , where  $n \geq 2$ . We denote by  $d$  the distance function on  $M$  and by  $i_M > 0$  the injectivity radius.

Let  $SM$  be the unit tangent bundle of  $M$  and  $d^S$  the Sasaki-distance function on  $SM$ . For  $v \in SM$  let  $\gamma_v : \mathbb{R} \rightarrow M$  be the unit speed geodesic such that  $\gamma_v'(0) = v$ . The geodesic flow  $\phi^t : SM \rightarrow SM$ ,  $t \in \mathbb{R}$ , acts on the compact metric space  $(SM, d^S)$  by diffeomorphisms, where  $\phi^t v = \gamma_v'(t)$ . For details and background we refer to [4].

A vector  $v \in SM$  is *periodic*, if there exists a  $t > 0$  such that  $\phi^t v = v$  and  $v$  is *recurrent* if for every  $\varepsilon > 0$  there exists  $s > 0$  such that  $d^S(\phi^s v, v) < \varepsilon$ . Denote by  $\mathcal{P}_\phi$  and  $\mathcal{R}_\phi$  the flow-invariant sets of periodic respectively of recurrent vectors. Thus if  $v \in \mathcal{R}_\phi$  then for a given  $t \in \mathbb{R}$ ,  $\varepsilon > 0$ , there exists  $s = s(t, \varepsilon)$  such that  $d^S(\phi^{t+s} v, \phi^t v) < \varepsilon$ .

We now adjust the definitions of  $F$ -aperiodic and  $\varphi$ -aperiodic points to the setting of the geodesic flow.

**Definition 4.1.** Let  $F : (0, \infty) \rightarrow [0, \infty)$  be a non-increasing function and  $s_0 > 0$  be a constant, called the *minimal shift*. A vector  $v \in SM$  is called *F-aperiodic* (with minimal shift  $s_0$ ) at  $t_0 \in \mathbb{R}$  if for every  $\varepsilon > 0$ , whenever

$$d^S(\phi^{t_0} v, \phi^{t_0+s} v) < \varepsilon$$

for some shift  $s > s_0$ , then  $s > F(\varepsilon)$ . If  $v$  is  $F$ -aperiodic at every time  $t_0$  then  $v$  is called *F-aperiodic* (with minimal shift  $s_0$ ).

Note that in contrast to the discrete setting in Section 2 (where  $s \in \mathbb{N}$ , i.e.  $s \geq 1$ ) we now have to specify the additional parameter  $s_0$ , since  $d^S(\phi^{t_0} v, \phi^{t_0+s} v) = s$  for  $s$  small enough.

We also have to generalize the notion of  $\varphi$ -aperiodicity. All geodesics will be assumed to be unit speed. Note that as in the case of the Bernoulli-shift, two vectors in the Sasaki-distance are very close if and only if the trajectories of the corresponding geodesics are close (in the Riemannian distance) to each other for a long time. Thus we may reformulate  $\varphi$ -aperiodicity in terms of the *fellow traveller length*.

Herefore we introduce a second parameter, the *distance constant*  $\varepsilon_0 > 0$ .

**Definition 4.2.** Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a non-decreasing function, let  $0 < \varepsilon_0 < i_M$  and  $s_0 \geq \varepsilon_0$ . A geodesic  $\gamma : \mathbb{R} \rightarrow M$  is called  *$\varphi$ -aperiodic* at time  $t_0 \in \mathbb{R}$  if for every length  $l > \varepsilon_0$ , whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0 \quad \text{for all } 0 \leq t \leq l$$

for some shift  $s > s_0$ , then  $s > \varphi(l)$ . If  $\gamma$  is  $\varphi$ -aperiodic at every time  $t_0$ , it is called  *$\varphi$ -aperiodic* (with parameters  $(s_0, \varepsilon_0)$ ).



The geodesic flow on compact hyperbolic manifolds is ergodic with respect to the Liouville measure  $\mu$  (on the Borel- $\sigma$ -algebra of  $SM$ ). A systole of  $M$  has length  $2i_M$  which equals the systolic period. For a non-decreasing function  $\varphi$  let  $\mathcal{F}_\phi$  be the set of  $\varphi$ -aperiodic geodesics (with respect to  $(s_0, \varepsilon_0)$ ), which is invariant under the geodesic flow  $\phi^t$ . Since  $\mu$  is positive on open sets, one can show as in Lemma 2.3, that the set  $\mathcal{F}_\phi$  is of zero  $\mu$ -measure if and only if  $\varphi$  is not bounded by either  $s_0$  or  $2i_M - \varepsilon_0$ .

The main result of this section is the following, which will be proved in the Section 5.

**Theorem 4.3.** *Assume that  $i_M > \ln(2)$  and let  $\varepsilon_0 > 0$  such that  $\ln(2) + \varepsilon_0 < i_M$ . Let*

$$\varphi_\delta(l) = e^{\delta(n-1)l},$$

where  $0 < \delta < 1$ . Then there exists a minimal length  $l_0 = l_0(\delta, i_M, n, \varepsilon_0)$  and a geodesic  $\gamma : \mathbb{R} \rightarrow M$  which satisfies for every  $t_0 \in \mathbb{R}$  and all  $l \geq l_0$ , whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0 \quad \text{for all } 0 \leq t \leq l \quad (4.1)$$

for some shift  $s > \varepsilon_0$ , then  $s > \varphi_\delta(l)$ .

Note that for  $\varepsilon_0 = i_M/2$ , if a geodesic  $\gamma : \mathbb{R} \rightarrow M$  satisfies (4.1) then  $R_\gamma(l) \geq \varphi_\delta(l)$  for all  $l \geq l_0$ , where  $R_\gamma$  is the recurrence time introduced in Paragraph 1. Theorem 1.2 is hence a corollary of Theorem 4.3.

*Proof of Theorem 1.2.* By (1.2) there exists for every  $\tau > 0$  some  $l_1 = l_1(\tau) \geq 0$  such that for all  $l \geq l_1$  we have

$$\varphi(l) \leq e^{(1+\tau)(n-1)\delta l}.$$

Since  $\delta < 1$  we let  $\tau_0 > 0$  such that  $\bar{\delta} := (1 + \tau_0)\delta < 1$ . From Theorem 4.3 for  $\varepsilon_0 = i_M/2$ , there exists an  $l_2 = l_2(\bar{\delta}, i_M, n)$  and a geodesic  $\gamma : \mathbb{R} \rightarrow M$  such that for every  $t_0 \in \mathbb{R}$  and  $l \geq l_2$ , whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \frac{i_M}{2} \quad \text{for all } 0 \leq t \leq l,$$

for some shift  $s > i_M/2$ , then  $s > e^{\bar{\delta}(n-1)l}$ . If we set  $l_0 := \max\{l_1, l_2\}$  then  $s > e^{\bar{\delta}(n-1)l} \geq \varphi(l)$  whenever  $l \geq l_0$  and the proof is finished.  $\square$

In order to prove Theorem 4.3 we discretize our geodesics. Therefore we need a third parameter, the *discretization constant*  $r_0 > 0$ . To a geodesic  $\gamma : \mathbb{R} \rightarrow M$  we consider the *discrete geodesic*

$$\bar{\gamma} : \mathbb{Z} \rightarrow M, \quad \bar{\gamma}(i) := \gamma(i \cdot r_0).$$

**Definition 4.4. (Discrete Definition)** Let  $\bar{\varphi} : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing function and let the parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$  be given where  $\bar{s}_0 \in \mathbb{N}_0$ ,  $0 < \bar{\varepsilon}_0 < i_M$  and  $0 < r_0 < \bar{\varepsilon}_0$ . A discrete geodesic  $\bar{\gamma} : \mathbb{Z} \rightarrow M$  is called  $\bar{\varphi}$ -aperiodic at time  $i \in \mathbb{Z}$  if for  $l \in \mathbb{N}$ , whenever

$$d(\bar{\gamma}(i + j), \bar{\gamma}(i + s + j)) < \bar{\varepsilon}_0 \quad \text{for all } j \in \{0, \dots, l\} \quad (4.2)$$

for some shift  $s > \bar{s}_0$ , then  $s > \bar{\varphi}(l)$ .  $\bar{\gamma}$  is called  $\bar{\varphi}$ -aperiodic (with parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ ) if it is  $\bar{\varphi}$ -aperiodic at every time  $i \in \mathbb{Z}$ .

Note that, given a  $\bar{\varphi}$ -aperiodic geodesic  $\bar{\gamma} : \mathbb{Z} \rightarrow M$  (with the parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ ), the corresponding geodesic  $\gamma : \mathbb{R} \rightarrow M$  is continuously  $\varphi$ -aperiodic in the following way.

**Lemma 4.5.** *For a non-decreasing function  $\bar{\varphi} : [0, \infty) \rightarrow [0, \infty)$  and the parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$  let  $\bar{\gamma} : \mathbb{Z} \rightarrow M$  be a  $\bar{\varphi}|_{\mathbb{N}_0}$ -aperiodic geodesic. For  $r_0 \leq l \in \mathbb{R}$ , define*

$$\varphi(l) := r_0 \cdot \bar{\varphi}\left(\frac{l - r_0}{r_0}\right) - r_0.$$

*Then  $\gamma$  is  $\varphi$ -aperiodic with respect to the minimal shift  $s_0 = (\bar{s}_0 + 1)r_0$  and the distance constant  $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 > 0$ .*

*Conversely, if  $\gamma : \mathbb{R} \rightarrow M$  is  $\varphi$ -aperiodic with parameters  $(s_0, \varepsilon_0)$  then for  $r_0 < \varepsilon_0$ , let*

$$\bar{\varphi}(l) := \varphi(l \cdot r_0) / r_0.$$

*Then  $\bar{\gamma} : \mathbb{Z} \rightarrow M$  is  $\bar{\varphi}$ -aperiodic with parameters  $(\lceil s_0/r_0 \rceil, \varepsilon_0, r_0)$ .*

*Proof.* For  $t_0 \in \mathbb{R}$ ,  $L \geq r_0$  and  $s > (\bar{s}_0 + 1)r_0$  assume that  $d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0$  for all  $0 \leq t \leq L$ . If we set  $i := \lceil \frac{t_0}{r_0} \rceil$  and  $i + \bar{s} := \lceil \frac{t_0 + s}{r_0} \rceil$  whereas  $l := \lfloor \frac{L}{r_0} \rfloor$ , we have  $i, l \geq 1$  and  $\bar{s} > \bar{s}_0$ . Then, since  $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 < i_M$  and the distance function is locally convex, one checks by the triangle inequality that  $d(\bar{\gamma}(i), \bar{\gamma}(i + \bar{s})) < \bar{\varepsilon}_0$  and  $d(\bar{\gamma}(i + l), \bar{\gamma}(i + \bar{s} + l)) < \bar{\varepsilon}_0$ . In particular,  $d(\bar{\gamma}(i + j), \bar{\gamma}(i + \bar{s} + j)) < \bar{\varepsilon}_0$  for all  $0 \leq j \leq l$ . Thus,  $\bar{s} > \bar{\varphi}(l)$  so that

$$s \geq (\bar{s} - 1)r_0 > (\bar{\varphi}(l) - 1)r_0 \geq \left(\bar{\varphi}\left(\frac{L}{r_0} - 1\right) - 1\right)r_0 = \varphi(L)$$

since  $(l + 1)r_0 \geq L$ . This finishes the first part of the Lemma. The second part follows analogously.  $\square$

In terms of Lemma 4.5 we are left with stating the existence theorem for discrete  $\bar{\varphi}$ -aperiodic geodesics. Recall that for an unbounded function  $\bar{\varphi}$  we defined its discrete right-inverse  $\bar{\ell} : \mathbb{N} \rightarrow \mathbb{N}_0$  in (3.4) which is also non-decreasing and unbounded.

**Theorem 4.6.** *Let  $\bar{\varphi} : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing, unbounded function. Assume that  $\ln(2) < r_0 < \bar{\varepsilon}_0 < i_M$  and  $\bar{s}_0 \in \mathbb{N}_0$  such that for all  $l \geq \bar{s}_0$ ,*

$$\lfloor \bar{\varphi}(l) \rfloor > l, \quad \text{and} \quad \bar{\ell}(\bar{s}_0) \geq 1, \quad (4.3)$$

*and moreover, that there exists a constant  $c \in (1, 2^{n-1})$  such that*

$$2^{n-1} - \bar{c} \cdot \sum_{l=\bar{\ell}(\bar{s}_0)}^{\infty} \frac{\lfloor \bar{\varphi}(l) \rfloor - \lfloor \bar{\varphi}(l-1) \rfloor}{c^l} \geq c, \quad (4.4)$$

*where  $\bar{c}$  is an explicit constant depending only on  $n$  and  $i_M$ . Then there exist a  $\bar{\varphi}$ -aperiodic geodesic  $\gamma : \mathbb{Z} \rightarrow M$  with the parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ .*

*Remark.* Since  $\bar{\ell}$  is unbounded, condition (4.4) depends again essentially on the convergence of the sum in (4.4). For instance, let  $\delta \in (0, 1)$  and define  $\bar{\varphi}(l) = 2^{\delta(n-1)l}$  and let  $c \in (2^{\delta(n-1)}, 2^{n-1})$ . Then, since  $\bar{\ell}(s) = \lceil \frac{1}{\delta(n-1)\ln(2)} \ln(s) \rceil$  for  $s \geq 0$ , there exists a minimal shift  $\bar{s}_0 = \bar{s}_0(n, \delta, \bar{c}, c)$  such that (4.3) and (4.4) are satisfied.

The constant  $\bar{c}$  of condition (4.4) can in fact be sharpened to be also dependent on  $\bar{s}_0$ , in which case it is strictly decreasing in  $\bar{s}_0$ . It will be explicitly defined in the proof of claim 5.12. We may give a rough upper bound of  $\bar{c}$  which is independent of  $\bar{s}_0$  by

$$\bar{c} \leq \lceil (3 \cosh(i_M) \sqrt{n+1})^{n-1} \rceil \left\lceil \frac{\int_0^{5i_M + 4\ln(\sqrt{n+1}/2)} \sinh(t)^{n-1} dt}{\int_0^{i_M/2} \sinh(t)^{n-1} dt} \right\rceil. \quad (4.5)$$

The lower bound  $\ln(2)$  on the injectivity radius is necessary for the proof. However we believe that the result should be valid without this bound. Moreover, a version of Theorem

4.6 remains true for  $M$  a closed  $n$ -dimensional Riemannian manifold of negative sectional curvature.

*Remark.* Again, the critical function  $\varphi$  such that  $\varphi$ -aperiodic geodesics might or might not exist seems to be the function  $\varphi(s) = e^{(n-1)s}$  and the critical exponent  $n - 1$  equals the topological entropy of  $(SM, \phi^t)$ .

Lemma 2.2 gives an upper bound on the growth rate of non-increasing functions  $F : (0, \infty) \rightarrow (0, \infty)$  for which  $F$ -aperiodic geodesics can exist. In fact, since  $SM$  is a  $(2n - 1)$ -dimensional manifold, its box dimension is  $2n - 1$ . Discretizing  $\phi^t$  by the time  $t_0$ -map  $\phi^{t_0}$  where  $t_0 = t_0(i_M) > 0$  is sufficiently small, gives the upper bound

$$\limsup_{\varepsilon \rightarrow 0} \frac{\ln(F(\varepsilon))}{\ln(2/\varepsilon)} \leq 2n - 1.$$

*Remark.* For a closed geodesic  $\alpha : \mathbb{R} \rightarrow M$ , let  $\mathcal{N}_{\varepsilon_0}(\alpha)$  be the (closed)  $\varepsilon_0/2$ -neighborhood of  $\alpha$  in  $M$ , where  $\varepsilon_0 > 0$  sufficiently small. When a geodesic  $\gamma : \mathbb{R} \rightarrow M$  enters  $\mathcal{N}_{\varepsilon_0}(\alpha)$  at time  $t_0$  let  $\mathbf{p}_\alpha(\gamma, t_0)$  be the *penetration length* of  $\gamma$  in  $\alpha$  at time  $t_0$ , that is, the maximal length  $L \in [0, \infty]$  of an interval  $I$ ,  $t_0 \in I$ , such that  $\gamma(t) \in \mathcal{N}_{\varepsilon_0}(\alpha)$  for all  $t \in I$ . Set  $\mathbf{p}_\alpha(\gamma, t_0) = 0$  if  $\gamma(t_0) \notin \mathcal{N}_{\varepsilon_0}(\alpha)$ . Then by [10], for  $\mu$ -almost every  $v \in SM$  the limit

$$\limsup_{t \rightarrow \infty} \frac{\mathbf{p}(\gamma_v(t))}{\ln(t)} \quad (4.6)$$

exists and equals  $1/(n - 1)$ .

Moreover, the penetration length reflects the *depth* in which  $\gamma$  enters the neighborhood  $\mathcal{N}_{\varepsilon_0}(\alpha)$ . The study of depths or penetration lengths in an adequate convex set of negatively curved manifolds, such as the  $\varepsilon$ -neighborhood of totally geodesic embedded submanifold or the cusp-neighborhood of a finite-volume hyperbolic manifold, leads to the theory of diophantine approximation in negatively curved manifolds; see for instance [7, 9, 10, 13, 14, 15, 17, 18] to give only a short and incomplete list. In general, a sequence of depths or penetration lengths and times of  $\gamma$  in these convex sets reflects "how well  $\gamma$  is approximated", where  $\gamma$  is called *badly approximable* if any such sequence is bounded; see [9, 10].

Now, let  $\gamma$  be a  $\varphi$ -aperiodic geodesic ( $\varphi$  unbounded) with respect to the parameters  $s_0$  and  $\varepsilon_0$  and let  $\alpha$  be **any** closed geodesic in  $M$ . Then, it can be seen that the penetration lengths of  $\gamma$  in  $\mathcal{N}_{\varepsilon_0}(\alpha)$  are bounded by a constant depending only on  $\varphi$ ,  $\varepsilon_0$  and the length of  $\alpha$  (and  $s_0$  respectively). Therefore, the notion of  $\varphi$ -aperiodicity is linked to bad approximation; recall also Example 1. In particular, the limit of (4.6) equals 0 for  $\gamma$ .

## 5. PROOFS

Let  $\varphi : \mathbb{N}_0 \rightarrow [0, \infty)$  be a non-decreasing unbounded function. Recall the definition of the function  $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$  given by

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\},$$

see (3.4). The following properties hold:  $\ell$  is non-decreasing and for  $s$  and  $l \in \mathbb{N}_0$ , we have

$$\begin{aligned} \varphi(\ell(s)) &\geq s, \\ l < \ell(s) &\iff \varphi(l) < s, \\ l \geq \ell(s) &\iff \varphi(l) \geq s. \end{aligned} \quad (5.1)$$

*Proof.* For the first property, clearly  $\varphi(\min\{j : \varphi(j) \geq s\}) \geq s$ . Let  $l < \ell(s)$  and assume  $s \leq \varphi(l)$ . Then  $\ell(s) = \min\{j : \varphi(j) \geq s\} \leq l$ ; a contradiction. If  $s > \varphi(l)$  then  $\ell(s) = \min\{j : \varphi(j) \geq s\} > l$  and if  $\varphi(l) \geq s$  then  $\ell(j) = \min\{j : \varphi(j) \geq s\} \leq l$ . Also, if  $l \geq \ell(s)$  then  $\varphi(l) \geq \varphi(\ell(s)) \geq s$ .  $\square$

**5.1. Proof of Theorem 3.3.** Recall that  $\Sigma^+(m) = \{w : \{1, \dots, m\} \rightarrow \mathcal{A}\}$  is the set of words of length  $m$ . We consider  $\Sigma^+(m)$  to be a subset of  $\Sigma^+ = \mathcal{A}^{\mathbb{N}}$  (for example, by extending an element  $w \in \Sigma^+(m)$  to an element  $\bar{w} \in \Sigma^+$  by setting  $\bar{w}(i) = a$  for all  $i > m$ , where  $a \in \mathcal{A}$  is fixed).

**Definition 5.1.** Let  $m \in \mathbb{N}$ .  $w \in \Sigma^+(m)$  is called  $\varphi$ -aperiodic if for all  $i, s \in \mathbb{N}$  and  $l \in \mathbb{N}_0$  such that  $i + s + l \leq m$  whenever

$$[w(i) \dots w(i + l)] = [w(i + s) \dots w(i + s + l)]$$

we have  $s > \varphi(l)$ .

Let  $l_0 := \min\{j \in \mathbb{N}_0 \cup \{-1\} : \varphi(j + 1) \neq 0\}$  and note that  $\ell(s) > l_0$  for all  $s \in \mathbb{N}$ . For  $m \in \mathbb{N}$ , define the *admissible set* by

$$A(m) := \{(i, s) \in \mathbb{N} \times \mathbb{N} : i + s + \ell(s) = m\},$$

if  $m \geq m_0 := 2 + \ell(1) > 2 + l_0$  and let  $A(m)$  be empty for  $m < m_0$ . Then, for  $(i, s) \in A(m)$  where  $m \geq m_0$ , we define the sets

$$C_{is} := \{w \in \Sigma^+(m) : [w(i) \dots w(i + \ell(s))] \neq [w(i + s) \dots w(i + s + \ell(s))]\},$$

called *conditions*.

*Remark.* Note that  $s > \varphi(\ell(s) - 1)$  for  $\ell(s) > 0$  but  $s \leq \varphi(\ell(s))$ . Therefore  $\ell(s)$  determines the critical length of a given shift  $s$  with respect to  $\varphi$ .

For  $w \in \Sigma^+(m)$  and  $1 \leq n \leq m$  let  $w|_n := [w(1) \dots w(n)] \in \Sigma^+(n)$ . This leads to the reformulation of  $\varphi$ -aperiodic words:

**Lemma 5.2.** For  $m < m_0$  every word  $w \in \Sigma^+(m)$  is  $\varphi$ -aperiodic. For  $m \geq m_0$ , a word  $w \in \Sigma^+(m)$  is  $\varphi$ -aperiodic if and only if for all  $n \leq m$  and all  $(i, s) \in A(n)$  we have  $w|_n \in C_{is}$ .

*Proof.* First, let  $m < m_0$ . Then for every  $i, s \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$  such that  $i + s + l \leq m < 2 + \ell(1)$  we have in particular  $l < \ell(1)$ . Equivalently,  $\varphi(l) < 1$  so that  $s > \varphi(l)$  and every word  $[w(1) \dots w(m)]$  follows to be  $\varphi$ -aperiodic.

Now let  $m \geq m_0$ . Let  $w$  be  $\varphi$ -aperiodic and assume  $w|_n \notin C_{is}$  for some  $i$  and  $s$  in  $\mathbb{N}$  such that  $i + s + \ell(s) = n \leq m$ . Then

$$[w(i) \dots w(i + \ell(s))] = [w(i + s) \dots w(i + s + \ell(s))]$$

and by (3.1), we have  $s > \varphi(\ell(s))$ ; a contradiction to  $\varphi(\ell(s)) \geq s$ .

Conversely, assume that  $w$  is not  $\varphi$ -aperiodic. Then there are  $i, s \in \mathbb{N}$  and  $l \in \mathbb{N}_0$  such that  $i + s + l \leq m$  and

$$[w(i) \dots w(i + l)] = [w(i + s) \dots w(i + s + l)]$$

with  $s \leq \varphi(l)$ . This implies that  $\ell(s) \leq l$  and in particular

$$[w(i) \dots w(i + \ell(s))] = [w(i + s) \dots w(i + s + \ell(s))].$$

Hence, it follows that  $w|_n \notin C_{is}$  since  $i + s + \ell(s) = n \leq m$  so that  $(i, s) \in A(n)$ .  $\square$

Note that by the same arguments as in the previous proof, a word  $w \in \Sigma^+$  is  $\varphi$ -aperiodic if and only if for all  $n \geq m_0$  and all  $(i, s) \in A(n)$  we have  $w|_n \in C_{is}$ .

For  $m \in \mathbb{N}$  such that  $m \geq m_0$  the set of good words of length  $m$  is therefore given by

$$\mathcal{W}^g(m) = \{w \in \Sigma^+(m) : w|_n \in C_{is} \text{ for all } (i, s) \in A(n) \text{ where } n \leq m\},$$

and by  $\mathcal{W}^g(m) = \Sigma^+(m)$  otherwise. Let

$$\mathcal{C}_m = \{C_{is} : (i, s) \in A(m)\}$$

be the set of conditions at place  $m$  which is empty if and only if  $m < m_0$ . Clearly, if  $w \in \mathcal{W}^g(m)$  then  $w|_n \in \mathcal{W}^g(n)$  for  $n \leq m$ .

**Lemma 5.3.** *For  $m \in \mathbb{N}$ ,*

$$|\mathcal{W}^g(m+1)| \geq k \cdot |\mathcal{W}^g(m)| - \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{W}^g(i+s-1)|$$

*Proof.* If  $m+1 < m_0$  then  $\mathcal{C}_{m+1}$  is empty and the claim follows. Hence let  $m+1 \geq m_0$ . Set  $L = \{w \in \Sigma^+(m+1) : w|_m \in \mathcal{W}^g(m)\}$ . Then

$$\mathcal{W}^g(m+1) = L \cap \left( \bigcap_{C_{is} \in \mathcal{C}_{m+1}} C_{is} \right) = L \setminus \left( \bigcup_{C_{is} \in \mathcal{C}_{m+1}} (L \cap C_{is}^C) \right),$$

where  $C_{is}^C$  denotes the complement of  $C_{is}$ . Fix some condition  $C_{is} \in \mathcal{C}_{m+1}$ . Since  $|L| = k \cdot |\mathcal{W}^g(m)|$  the Lemma follows from the following claim.  $\square$

**Claim 5.4.**  $|L \cap C_{is}^C| \leq |\mathcal{W}^g(i+s-1)|$ .

*Proof.* If  $Q := \{w|_{i+s-1} \in \Sigma^+(i+s-1) : w \in L\}$  then clearly  $|Q| \leq |\mathcal{W}^g(i+s-1)|$ . Decompose  $L$  into  $L = \bigcup_{q \in Q} L_q$  where  $L_q = \{w \in L : w|_{i+s-1} = q\}$ . By definition, different elements in  $L_q$  have different subwords  $[w(i+s) \dots w(m+1)]$  and moreover

$$L \cap C_{is}^C = \{w \in L : [w(i) \dots w(i+\ell(s))] = [w(i+s) \dots w(m+1)]\}.$$

Hence, if  $s > \ell(s)$  then an element  $w$  of  $L_q$ , which is also in  $C_{is}^C$ , is uniquely determined by  $q$ , that means,  $w$  is of the form  $w|_{i+s-1} = q$  and

$$[w(i+s) \dots w(m+1)] = [q(i) \dots q(i+\ell(s))].$$

If  $s \leq \ell(s)$  then one inductively checks that a word  $w$  in  $L_q \cap C_{is}^C$  is of the form  $w|_{i+s-1} = q$ ,

$$\begin{aligned} [w(i+j)s \dots w(i+(j+1)s-1)] &= [w(i+(j-1)s) \dots w(i+js-1)] = \dots = \\ &= [w(i) \dots w(i+s-1)] = [q(i) \dots q(i+s-1)] \end{aligned}$$

for  $1 \leq j \leq j_0$  where  $j_0$  is the maximal  $j$  such that  $i+(j+1)s-1 \leq m+1$ , and

$$[w(i+(j_0+1)s) \dots w(m+1)] = [q(i) \dots q(m+1-(i+(j_0+1)s))],$$

if  $i+(j_0+1)s < m+1$ . Again,  $w$  is uniquely determined by  $q$ . Hence in both cases,  $|L_q \cap C_{is}^C| \leq 1$  and therefore

$$|L \cap C_{is}^C| \leq |Q| \leq |\mathcal{W}^g(i+s-1)|$$

which proves the claim.  $\square$

The above Lemma yields the following crucial estimate:

**Lemma 5.5.** *For  $m \in \mathbb{N}$ ,*

$$|\mathcal{W}^g(m+1)| \geq (k - \lfloor \varphi(0) \rfloor) |\mathcal{W}^g(m)| - \sum_{j=1}^m (\lfloor \varphi(j) \rfloor - \lfloor \varphi(j-1) \rfloor) |\mathcal{W}^g(m-j)|. \quad (5.2)$$

*Proof.* For  $0 \leq j \leq m$  let

$$H_j = \{C_{is} \in \mathcal{C}_{m+1} : i + s - 1 = m - j\}, \quad (5.3)$$

possibly empty. If  $C_{is} \in H_j$  then  $i + s + \ell(s) = m + 1$  and  $i + s - 1 = m - j$ ; hence  $\ell(s) = j$ . Therefore,  $|H_j| \leq |\{s : \ell(s) = j\}|$ . We have  $\ell(s) \leq j$  if and only if  $s \leq \varphi(j)$  and thus

$$|\{s : \ell(s) \leq j\}| = |\{s : s \leq \varphi(j)\}| = \lfloor \varphi(j) \rfloor.$$

For  $j \geq 1$  this implies that

$$\begin{aligned} |H_j| &\leq |\{s : \ell(s) = j\}| = |\{s : \ell(s) \leq j\} \setminus \{s : \ell(s) \leq j - 1\}| \\ &= \lfloor \varphi(j) \rfloor - \lfloor \varphi(j - 1) \rfloor. \end{aligned}$$

Moreover,

$$|\{s : \ell(s) = 0\}| = |\{s \in \mathbb{N}_0 : \varphi(0) \geq s\}| = \lfloor \varphi(0) \rfloor.$$

Lemma 5.3 concludes the proof.  $\square$

Finally we show the existence of a  $\varphi$ -aperiodic word in  $\Sigma^+$ .

**Lemma 5.6.** *If condition (3.2) is satisfied, then  $|\mathcal{W}^g(m)| \geq c^m$ . In particular, there exists a  $\varphi$ -aperiodic word in  $\Sigma^+$ .*

*Proof.* For  $m + 1 < m_0$  we have that  $|\mathcal{W}^g(m + 1)| = k^{m+1} \geq c^{m+1}$ . For  $m + 1 \geq m_0$  assume that  $|\mathcal{W}^g(n)| \geq c \cdot |\mathcal{W}^g(n - 1)|$  for all  $n \leq m$ . Then, by the previous Lemma,

$$\begin{aligned} |\mathcal{W}^g(m + 1)| &\geq (k - \lfloor \varphi(0) \rfloor) |\mathcal{W}^g(m)| - \sum_{j=1}^m (\lfloor \varphi(j) \rfloor - \lfloor \varphi(j - 1) \rfloor) |\mathcal{W}^g(m - j)| \\ &\geq (k - \lfloor \varphi(0) \rfloor) |\mathcal{W}^g(m)| - \sum_{j=1}^m \frac{\lfloor \varphi(j) \rfloor - \lfloor \varphi(j - 1) \rfloor}{c^j} |\mathcal{W}^g(m)| \\ &\geq \left( k - \lfloor \varphi(0) \rfloor - \sum_{j=1}^\infty \frac{\lfloor \varphi(j) \rfloor - \lfloor \varphi(j - 1) \rfloor}{c^j} \right) |\mathcal{W}^g(m)| \geq c \cdot |\mathcal{W}^g(m)|, \end{aligned} \quad (5.4)$$

where we used condition (3.2) in the last inequality. Now Lemma 5.2 implies the existence of a  $\varphi$ -aperiodic word in  $\Sigma^+$ .  $\square$

Given a  $\varphi$ -aperiodic word  $w \in \Sigma^+$  and a letter  $a \in \mathcal{A}$ , extend  $w$  to a word  $\dots aaaw =: \bar{w} \in \Sigma$  (in the obvious way). Consider the sequence  $\{T^n \bar{w}\}_{n \in \mathbb{N}}$  in the compact space  $\Sigma$  and let  $w_0$  be an accumulation point. Note that from the definition of the metric  $\bar{d}$ , a sequence  $w^n$  in  $\Sigma$  converges to a word  $w_0 \in \Sigma$  if and only if for every  $l \in \mathbb{N}_0$  there exists  $N \in \mathbb{N}$  such that  $[w^n(-l) \dots w^n(l)] = [w_0(-l) \dots w_0(l)]$  for every  $n \geq N$ . It therefore follows that  $\varphi$ -aperiodicity is a closed condition (as showed similarly in Lemma 2.4). Since every  $T^n \bar{w}$  is  $\varphi$ -aperiodic starting at time  $-(n - 1)$ ,  $w_0$  is a  $\varphi$ -aperiodic word in  $\Sigma$ . This proves Theorem 3.3.

**5.2. Proof of Theorem 4.6.** Recall that  $M$  is a closed hyperbolic manifold of dimension  $n \geq 2$  and we have  $\ln(2) < r_0 < \bar{\varepsilon}_0 < i_M$ . Moreover  $\bar{\varphi} : \mathbb{N}_0 \rightarrow [0, \infty)$  is a non-decreasing unbounded function for which conditions (4.3) and (4.4) are satisfied with respect to the given minimal shift  $\bar{s}_0 \in \mathbb{N}_0$ .

A reference for the following is given by [4, 19]. Let  $\mathbb{H}^n$  be the  $n$ -dimensional hyperbolic upper half-space model where  $d$  denotes the hyperbolic distance function on  $\mathbb{H}^n$ . Let  $\Gamma$  be the discrete, torsion-free subgroup of the isometry group of  $\mathbb{H}^n$  identified with the fundamental group  $\pi_1(M)$  of  $M$  acting cocompactly on  $\mathbb{H}^n$  such that the manifold  $\Gamma \backslash \mathbb{H}^n$  with the induced smooth and metric structure is isometric to  $M$ . Let  $\pi : \mathbb{H}^n \rightarrow \Gamma \backslash \mathbb{H}^n \cong M$  be the projection map. Assume all geodesic segments, rays or lines to be parametrized by arc length and identify their images with their point sets in  $\mathbb{H}^n$ . Let  $\partial_\infty \mathbb{H}^n$  be the set of

equivalence classes of asymptotic rays in  $\mathbb{H}^n$  which we identify with the set  $\mathbb{R}^{n-1} \cup \{\infty\}$ , where  $\mathbb{H}^n - \{\infty\} = \mathbb{H}^n \cup \mathbb{R}^{n-1}$  is equipped with the induced Euclidean topology. If  $\gamma$  is a ray in  $\mathbb{H}^n$  we will simply write  $\gamma(\infty)$  for the corresponding point in  $\partial_\infty \mathbb{H}^n$ . For any two points  $p$  and  $q$  in  $\bar{\mathbb{H}}^n$  denote by  $[p, q]$  the geodesic segment, ray or line in  $\mathbb{H}^n$  - depending on if  $p, q \in \mathbb{H}^n$ ,  $p \in \mathbb{H}^n$  and  $q \in \partial_\infty \mathbb{H}^n$ , or  $p, q \in \partial_\infty \mathbb{H}^n$  respectively - connecting  $p$  and  $q$ .

For  $t \in \mathbb{R}$  let  $H_t := \mathbb{R}^{n-1} \times \{e^{-t}\} \subset \mathbb{H}^n$ . This equals the horosphere based at  $\infty$  through the point  $\gamma(t)$  of the unit speed geodesic  $\gamma(t) = (0, e^{-t})$ . Let  $h_t$  be the induced length metric on  $H_t$  with respect to  $d$ . The geometry of horospheres in the hyperbolic space is well-known; see for instance [8] for the following facts.  $(H_t, h_t)$  is a complete and flat metric space, isometric to the  $(n-1)$ -dimensional Euclidean space. If  $\gamma_i : \mathbb{R} \rightarrow \mathbb{H}^n$  with  $\gamma_i(0) \in H_0$ ,  $i = 1, 2$ , are two geodesic lines in  $\mathbb{H}^n$  with  $\gamma_1(-\infty) = \gamma_2(-\infty) = \infty$  and  $\gamma_1(0), \gamma_2(0)$  in the same horosphere, let  $\mu(t) := h_t(\gamma_1(t), \gamma_2(t))$ . Then, for  $t \geq 0$ ,

$$\mu(t) = e^t \mu(0). \quad (5.5)$$

Moreover, for two points  $p, q$  in the same horosphere  $H_t$  we have

$$h_t(p, q) = 2 \sinh(d(p, q)/2). \quad (5.6)$$

Now let  $\tau > 0$  such that the discretization constant satisfies  $r_0 = \ln 2 + \tau$ . Let  $R > 0$  be a fixed length, say  $R = 1$ . Define  $Q$  to be an isometric copy of a closed  $(n-1)$ -dimensional cube  $[-R/2, R/2]^{n-1}$  of edge lengths  $R$  in the Euclidean space  $\mathbb{E}^{n-1}$  and contained in the horosphere  $H_0$ . Starting with the cube  $Q$  as a reference, we inductively shed shadows in the horospheres  $H_{mr_0}$ ,  $m \in \mathbb{N}$ , as follows:

**Definition 5.7.** Given two disjoint sets  $S$  and  $S'$  in  $\bar{\mathbb{H}}^n$ , the set  $\mathcal{S}(S; S') := \{q \in S' : S \cap [\infty, q] \neq \emptyset\}$  is called the *shadow of  $S$  in  $S'$*  (with respect to  $\infty$ ).

By (5.5), the shadow  $\mathcal{S}(Q; H_{r_0})$  of  $Q$  is an isometric copy of a closed  $(n-1)$ -dimensional cube of edge lengths  $e^{r_0}R = (2 + e^\tau)R$ , contained in  $H_{r_0}$ . Hence, there exist  $2^{n-1}$  disjoint isometric copies  $Q_j$ ,  $j \in \{1, \dots, 2^{n-1}\}$ , of  $Q$  in  $\mathcal{S}(Q; H_{r_0})$ ; see Figure 5.2.

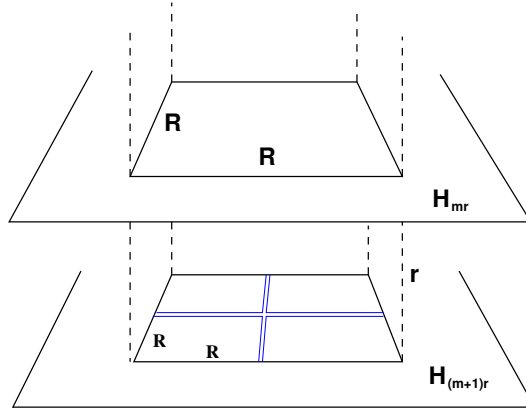


Figure 5.2:  $n = 3$ .

For  $m \geq 1$ , let the closed disjoint cubes  $Q_{i_1 \dots i_m}$  in  $H_{mr_0}$  be already defined. Fix a *cube*  $Q_{i_1 \dots i_m}$ , then, as above, the shadow

$$\mathcal{S}(Q_{i_1 \dots i_m}; H_{(m+1)r_0}) \subset H_{(m+1)r_0}$$

contains  $2^{n-1}$  disjoint isometric copies  $Q_{i_1 \dots i_m j}$  of  $Q$ ,  $j \in \{1, \dots, 2^{n-1}\}$ . Hence, for an alphabet  $\mathcal{A} = \{1, \dots, 2^{n-1}\}$ , we associate a finite word  $[w(1) \dots w(m+1)] \in \Sigma^+(m+1)$

to the cube  $Q_{i_1 \dots i_{m+1}}$  in  $H_{(m+1)r_0}$  where  $w(n) = i_n$  for all  $n \in \{1, \dots, m+1\}$ . In particular, we obtain a bijection of finite words  $\Sigma^+(m)$  of length  $m$  with the set of cubes

$$\mathcal{Q}(m) := \{Q_{i_1 \dots i_m} \subset H_{mr_0} : i_n \in \{1, \dots, 2^{n-1}\} \text{ for } 1 \leq n \leq m\}.$$

We denote the closed cubes  $Q_{i_1 \dots i_m}$  obtained in this way by  $q(1) \dots q(m)$  where  $q(n) \in \{1, \dots, 2^{n-1}\}$  for  $n \in \{1, \dots, m\}$ . Every sequence of cubes  $\{q(1)q(2) \dots q(m)\}_{m \in \mathbb{N}}$ , successively shadowed from the previous ones, determines a unique point

$$\eta := \bigcap_{m \in \mathbb{N}} \mathcal{S}(q(1) \dots q(m); \mathbb{R}^{n-1}) \in \mathbb{R}^{n-1},$$

since  $\mathcal{S}(q(1) \dots q(m); \mathbb{R}^{n-1})$ ,  $m \in \mathbb{N}$ , is a sequence of closed nested subsets of  $\mathbb{R}^{n-1}$  with diameters converging to 0. Define  $\eta =: q(1)q(2) \dots$  in  $\mathbb{R}^{n-1}$ . By construction, the geodesic line  $[\infty, \eta]$  runs through every cube  $q(1) \dots q(m)$ ,  $m \in \mathbb{N}$ , of the particular sequence. Hence, we obtain a bijection of infinite sequences  $q(1)q(2) \dots$  of cubes and words  $w =: [w(1)w(2) \dots]$  in  $\Sigma^+$ .

*Notation.* Given a cube  $q(1) \dots q(m)$  in  $\mathcal{Q}(m)$  and an integer  $n \leq m$ , let  $q(1) \dots q(m)|_n \in \mathcal{Q}(n)$  be the unique cube such that  $q(1) \dots q(m)$  lies in the shadow of  $q(1) \dots q(m)|_n$ . Moreover, for  $\xi \in \mathbb{R}^n$  we denote the geodesic subsegment  $[i, j](\xi)$  by

$$[i, j](\xi) := [\infty, \xi]|_{[ir_0, jr_0]} : [ir_0, jr_0] \rightarrow \mathbb{H}^n,$$

where we assume that  $[\infty, \xi](0) \in H_0$  and that  $i, j \in \mathbb{N}_0$  with  $i \leq j$ , which connects the horospheres  $H_{ir_0}$  to  $H_{jr_0}$  and is orthogonal to both. If  $i = j$ , then we write  $[i](\xi) := [i, i](\xi)$  which is the orthogonal projection of  $\xi$  on the horosphere  $H_{ir_0}$ .

We again define the *admissible set*

$$A(m) := \{(i, s) \in \mathbb{N} \times \mathbb{N} : i + s + \bar{\ell}(s) = m, s > \bar{s}_0\},$$

if  $m \geq m_0 := 2 + \bar{s}_0 + \bar{\ell}(\bar{s}_0 + 1)$  and set  $A(m)$  to be empty for  $m < m_0$ .

**Definition 5.8.** Let  $\psi \in \Gamma$  be an isometry and let  $i, s \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$ . If  $\xi \in \mathbb{R}^{n-1}$  such that  $d(\psi([i](\xi)), [i + s](\xi)) < \bar{\varepsilon}_0$  and also  $d(\psi([i + l](\xi)), [i + s + l](\xi)) < \bar{\varepsilon}_0$  we write

$$\psi([i, i + l](\xi)) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\xi).$$

In particular, by convexity of the distance function, we have for all  $j \in \{0, \dots, l\}$ ,

$$d(\psi([i, i + j](\xi)), [i + s, i + s + j](\xi)) < \bar{\varepsilon}_0. \quad (5.7)$$

We are now able to translate the proof of Theorem 3.3 for the existence of  $\varphi$ -aperiodic words into the existence of  $\varphi$ -aperiodic geodesics by counting good cubes:

**Definition 5.9.** Let  $m \in \mathbb{N}$ . A cube  $q(1) \dots q(m)$  in  $\mathcal{Q}(m)$  is called *good* if for every  $\xi \in \mathcal{S}(q(1) \dots q(m); \mathbb{R}^{n-1})$ , every  $\psi \in \Gamma$  and every  $i \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$ , whenever

$$\psi([i, i + l](\xi)) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\xi) \quad (5.8)$$

for some shift  $s > \bar{s}_0$  such that  $i + s + l \leq m$ , then  $s > \bar{\varphi}(l)$ . Otherwise  $q(1) \dots q(m)$  is called *bad*.

If the cube  $q(1) \dots q(m)$  is good, then, since  $\bar{\varepsilon}_0 < i_M$ , for every  $x \in q(1) \dots q(m)$  the projection of the geodesic segment  $[\infty, x]|_{[r_0, mr_0]}$  into  $M$  is  $\bar{\varphi}$ -aperiodic, up to length  $mr_0$ , with respect to condition (4.2) (see the proof Lemma 5.10 (2)).



Analogously to the proof of Theorem 3.3, for  $(i, s) \in A(m)$  and  $m \geq m_0$ , define

$$C_{is} := \{q(1) \dots q(m) \in \mathcal{Q}(m) : \text{for all } \xi \in \mathcal{S}(q(1) \dots q(m); \mathbb{R}^{n-1}) \text{ and } \psi \in \Gamma, \\ \psi([i, i + \bar{\ell}(s)](\xi)) \not\sim_{\bar{\varepsilon}_0} [i + s, m](\xi)\}$$

and let  $\mathcal{C}_m$  be the set of all  $C_{ij}$  for  $(i, j) \in A(m)$ . Note that  $\mathcal{C}_m$  is empty if  $m < m_0$ .

With respect to these definitions, the relationship between Definitions 4.4 and 5.9 respectively and the sets  $C_{is}$  is given by the following Lemma:

**Lemma 5.10.** (1) *For  $m < m_0$  every cube  $q(1) \dots q(m) \in \mathcal{Q}(m)$  is good. For  $m \geq m_0$ , the cube  $q(1) \dots q(m) \in \mathcal{Q}(m)$  is good if  $q(1) \dots q(m)|_n \in C_{is}$  for all  $n \leq m$  and  $(i, s) \in A(n)$ .*

(2) *Let  $q(1)q(2) \dots$  be an infinite sequence of cubes and let  $\eta \in \mathbb{R}^{n-1}$  be the unique corresponding limit point. The discrete geodesic  $\pi \circ [r_0, \infty)(\eta)$  in  $M$  is  $\bar{\varphi}$ -aperiodic at every time  $i \in \mathbb{N}$  if for all  $m \in \mathbb{N}$  and  $(i, s) \in A(m)$  the cube  $q(1) \dots q(m)$  in  $\mathcal{Q}(m)$  of the sequence  $q(1)q(2) \dots$  belongs to  $C_{is}$ .*

*Proof.* For (1), let first  $m < m_0$ . Let  $i, s \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$  such that  $s > \bar{s}_0$  and  $i + s + l \leq m < 2 + \bar{s}_0 + \bar{\ell}(\bar{s}_0 + 1)$ . In particular,  $l < \bar{\ell}(\bar{s}_0 + 1)$  so that  $\varphi(l) < \bar{s}_0 + 1 \leq s$  and every cube  $q(1) \dots q(m)$  follows to be good.

Now let  $m \geq m_0$ . Assume by absurd that  $q(1) \dots q(m)$  is not good and let  $\xi \in \mathcal{S}(q(1) \dots q(m); \mathbb{R}^{n-1})$  and  $\psi \in \Gamma$  such that for some  $i \in \mathbb{N}$ ,  $l \in \mathbb{N}_0$ , we have

$$\psi([i, i + l](\xi)) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\xi),$$

where  $s > \bar{s}_0$  with  $i + s + l \leq m$  and  $s \leq \bar{\varphi}(l)$ . Hence,  $\bar{\ell}(s) \leq l$  and for  $n := i + s + \bar{\ell}(s)$  we have in particular by (5.7),

$$\psi([i, i + \bar{\ell}(s)](\xi)) \sim_{\bar{\varepsilon}_0} [i + s, n](\xi).$$

Hence, we see that  $q(1) \dots q(m)|_n \notin C_{is}$  where  $(i, s) \in A(n)$  for  $n \leq m$ ; a contradiction.

For (2), assume that  $\bar{\gamma} := \pi \circ [r_0, \infty)(\eta)$  is not  $\bar{\varphi}$ -aperiodic at time  $i \in \mathbb{N}$ . Then there must be a shift  $s \in \mathbb{N}$  with  $s > \bar{s}_0$ , and  $l \in \mathbb{N}_0$  such that

$$d(\bar{\gamma}(i + j), \bar{\gamma}(i + s + j)) < \bar{\varepsilon}_0 \quad \text{for all } j \in \{0, \dots, l\},$$

where  $s \leq \bar{\varphi}(l)$ . Since  $\bar{\varepsilon}_0 < i_M$  and the distance function is convex, we also have  $d(\gamma((i + t)r_0), \gamma((i + s + t)r_0)) < \bar{\varepsilon}_0$  for all  $0 \leq t \leq l$  for the corresponding extended geodesic  $\gamma : \mathbb{R} \rightarrow M$ . By discreteness of  $\Gamma$ , there exist finitely many isometries  $\psi_1, \dots, \psi_q \in \Gamma$  and a subdivision of the interval  $[ir_0, (i + l)r_0]$  into  $[l_0r_0, l_1r_0], [l_1r_0, l_2r_0], \dots, [l_{q-1}r_0, l_qr_0]$  where  $l_0 = i$  and  $l_q = i + l$  and  $l_j \in \mathbb{R}$ , such that (with analogous notation as above)

$$\psi_{j+1}([l_j, l_{j+1}](\eta)) \sim_{\bar{\varepsilon}_0} [s + l_j, s + l_{j+1}](\eta), \quad j = 0, \dots, q - 1.$$

We thus have  $d(\psi_{j+1}([l_j, l_{j+1}](\eta)), [s + l_{j+1}](\eta)) < \bar{\varepsilon}_0$  and  $d(\psi_{j+2}([l_{j+1}, l_{j+2}](\eta)), [s + l_{j+1}](\eta)) < \bar{\varepsilon}_0$ . Since  $\bar{\varepsilon}_0 < i_M$  and every orbit of  $\Gamma$  is  $2i_M$ -separated (that is, for  $\psi, \bar{\psi} \in \Gamma$  we have  $d(\psi x, \bar{\psi} x) \geq 2i_M$  for any  $x \in \mathbb{H}^n$ ) it follows from the triangle inequality that  $\psi_{j+1}([l_j, l_{j+1}](\eta)) = \psi_{j+2}([l_{j+1}, l_{j+2}](\eta))$ ; hence  $\psi_{j+1} = \psi_{j+2}$  for all  $j = 0, \dots, q - 2$  since  $\Gamma$  acts freely. Therefore, we have an isometry  $\psi \in \Gamma$  such that

$$\psi([i, i + l](\eta)) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\eta)$$

where  $s \leq \bar{\varphi}(l)$ . The proof is now finished analogously to the case of (1).  $\square$

In view of Lemma 5.10, let for  $m \geq m_0$ ,

$\mathcal{Q}^g(m) = \{q(1) \dots q(m) \in \mathcal{Q}(m) : q(1) \dots q(m)|_n \in C_{is} \text{ for all } (i, s) \in A(n), n \leq m\}$ ,  
and  $\mathcal{Q}^g(m) = \mathcal{Q}(m)$  for  $m < m_0$ , which is a subset of all good cubes at step  $m$ .

**Lemma 5.11.** *Assume that condition (4.3) is satisfied. Then, for  $m \in \mathbb{N}$ ,*

$$|\mathcal{Q}^g(m+1)| \geq k|\mathcal{Q}^g(m)| - \bar{c} \cdot \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{Q}^g(i+s-1)|, \quad (5.9)$$

where  $\bar{c}$  is a constant depending only on  $n$ ,  $i_M$  and  $\bar{s}_0$ , and is strictly decreasing in  $\bar{s}_0$ .

*Proof.* If  $m+1 < m_0$  then  $\mathcal{C}_{m+1}$  is empty and the claim follows. Hence assume  $m+1 \geq m_0$ . Let

$$L = \{q(1) \dots q(m+1) \in \mathcal{Q}(m+1) : q(1) \dots q(m+1)|_m \in \mathcal{Q}^g(m)\}$$

and note that  $|L| = k|\mathcal{Q}^g(m)|$ . Then

$$\mathcal{Q}^g(m+1) = L \cap \left( \bigcap_{C_{is} \in \mathcal{C}_{m+1}} C_{is} \right) = L \setminus \left( \bigcup_{C_{is} \in \mathcal{C}_{m+1}} (L \cap C_{is}^C) \right),$$

where  $C_{is}^C$  is the complement of  $C_{is}$ . Fix some  $C = C_{is} \in \mathcal{C}_{m+1}$ . Define

$$Q = \{q(1) \dots q(m+1)|_{i+s-1} \in \mathcal{Q}(i+s-1) : q(1) \dots q(m+1) \in L\},$$

One checks that  $|Q| \leq |\mathcal{Q}^g(i+s-1)|$ . Let  $L = \bigcup_{q \in Q} L_q$  where

$$L_q = \{q(1) \dots q(m+1) \in L : q(1) \dots q(m)|_{i+s-1} = q\}.$$

It remains to show that each  $L_q \cap C^C$  contains at most  $\bar{c}$  cubes; in this case,

$$|L \cap C^C| \leq \bar{c} \cdot |Q| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|.$$

The following claim concludes the proof. □

**Claim 5.12.**  $|L_q \cap C^C| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|$ .

For the proof of the claim note that if (4.4) is satisfied, then for all  $l \geq \bar{s}_0$ ,

$$\lfloor \bar{\varphi}(l) \rfloor > l,$$

which implies that for all  $s > \bar{s}_0$ ,

$$\bar{\ell}(s) < s. \quad (5.10)$$

To see this, assume  $\bar{\ell}(s) \geq s$  for some  $s > \bar{s}_0$ . Then, by definition of  $\bar{\ell}$ ,  $\bar{\varphi}(j) < s$  for all  $s > j \in \mathbb{N}_0$ . In particular, for  $\bar{s}_0 < s$  we have  $\bar{\varphi}(\bar{s}_0) \geq \lfloor \bar{\varphi}(\bar{s}_0) \rfloor$ ; a contradiction to  $\lfloor \bar{\varphi}(\bar{s}_0) \rfloor > \bar{s}_0$ .

*Proof of the Claim 5.12.*  $L_q$  consists of cubes of the form  $q \cdot q(i+s) \dots q(m+1) \in \mathcal{Q}(m+1)$ . Hence, consider the point set  $W$  of all geodesic segments  $[i, i + \bar{\ell}(s)](\xi)$  where  $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$ ; see Figure 5.2. Since  $s > \bar{s}_0$  we have  $\bar{\ell}(s) < s$  by (5.10), and therefore  $s - 1 - \bar{\ell}(s) \geq 0$ . Moreover, by definition, the cube  $q$  in  $H_{(i+s-1)r_0}$  has  $h$ -edge lengths  $R$ . Thus from (5.5), the subset  $H_{i+\bar{\ell}(s)} \cap W$  is isometric to an Euclidean cube with  $h$ -edge length

$$e^{-(i+s-1)r_0 + (i+\bar{\ell}(s))r_0} R = e^{-(s-1-\bar{\ell}(s))r_0} R \leq R.$$

Since an Euclidean cube in  $\mathbb{E}^{n-1}$  of edge length  $L$  has diameter at most  $\sqrt{n-1}L$ , we obtain from (5.6) that the  $d$ -diameter of  $H_{i+\bar{\ell}(s)} \cap W$  is bounded above by

$$2 \operatorname{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0} \sqrt{n-1} R / 2). \quad (5.11)$$

In the same way, the  $h$ -edge length of  $H_{ir_0} \cap W$  is given by

$$e^{-(s-1)r_0} R. \quad (5.12)$$

Now, by definition, for every  $q \cdot q(i+s) \dots q(m+1) \in L_q \cap C^C$  there exists  $\psi \in \Gamma$  such that  $\psi([i, i+\bar{\ell}(s)](\xi)) \sim_{\bar{\varepsilon}_0} [i+s, m+1](\xi)$  for some  $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$ . In particular,  $x := [m+1](\xi)$  must belong to the  $\bar{\varepsilon}_0$ -neighborhood of  $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ . Thus, we want to estimate the maximal number of cubes in  $\mathcal{Q}(m+1)$  which intersect with the  $\bar{\varepsilon}_0$ -neighborhood of  $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ . Let therefore also  $y \in H_{(m+1)r_0}$  belong to the  $\bar{\varepsilon}_0$ -neighborhood of  $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ . By the triangle inequality and by (5.11), we have

$$d(x, y) \leq 2\bar{\varepsilon}_0 + 2 \operatorname{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0} \sqrt{n-1} R/2).$$

Therefore, again from (5.6), the  $h$ -diameter of the intersection of the  $\bar{\varepsilon}_0$ -neighborhood of  $\psi(W \cap H_{i+s+\bar{\ell}(s)})$  with  $H_{(m+1)r_0}$  is bounded above by

$$\bar{r}_1(s) := 2 \sinh(\bar{\varepsilon}_0 + \operatorname{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0} \sqrt{n-1} R/2)).$$

On the other hand, the cubes  $q \cdot q(i+s) \dots q(m+1) \in \mathcal{Q}(m+1)$  are disjoint and have Euclidean volume  $R^{n-1}$ . Therefore, we set

$$\bar{c}_1(s) := \lceil \frac{(\bar{r}_1(s) + \sqrt{n-1} R)^{n-1}}{R^{n-1}} \rceil.$$

Hence, the  $\bar{\varepsilon}_0$ -neighborhood of  $\psi(W \cap H_{i+s+\bar{\ell}(s)})$  can intersect at most  $\bar{c}_1(s)$  cubes in  $\mathcal{Q}(m+1)$ . Since  $q(1) \dots q(m)$  is good for every  $q(1) \dots q(m+1) \in L_q$ , we conclude that, with respect to  $\psi$ , at most  $\bar{c}_1(s)$  cubes can become bad in  $L_q \cap C^C$ .

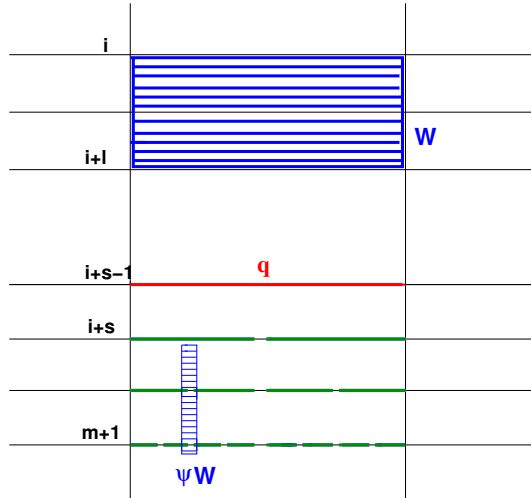


Figure 5.2:  $n = 2$ .

Now, let  $\bar{y}$  be the center of  $W \cap H_{ir_0}$ , which is isometric to a cube in the Euclidean space of edge length  $e^{-(s-1)r_0} R$  by (5.12) and contained in the cube  $q|_i$ . From (5.6),  $W \cap H_{ir_0}$  must be contained in the hyperbolic ball  $B_d(\bar{y}, \bar{r}_2(s))$ , where

$$\bar{r}_2(s) = 2 \operatorname{arcsinh}(e^{-(s-1)r_0} \sqrt{n-1} R/4).$$

Note that if there is some point  $p \in W \cap H_{ir_0}$  and some  $\psi \in \Gamma$  such that  $d(\psi p, \bar{q}) < \bar{\varepsilon}_0$ , where  $\bar{q} := \mathcal{S}(q, H_{(i+s)r_0})$ , then  $d(\psi \bar{y}, \bar{q}) < \bar{\varepsilon}_0 + \bar{r}_2(s)$ . In particular, for every cube  $q \cdot q(i+s) \dots q(m+1) \in L_q \cap C^C$  there exists such an isometry  $\psi$ . But since the orbit  $\Gamma \bar{y}$  is  $2i_M$ -separated, the open metric balls  $B(\psi \bar{y}, i_M)$ ,  $\psi \in \Gamma$ , are disjoint and there can only be finitely many, say  $\bar{c}_2(j)$ , intersecting the  $\max\{\bar{\varepsilon}_0 + \bar{r}_2(s) - i_M, 0\}$ -neighborhood of  $\bar{q}$ .

In fact, from (5.5) and (5.6), the  $h$ -diameter of  $\bar{q}$  is bounded above by  $e^{r_0}\sqrt{n-1}R$  and  $\bar{q}$  must be contained in a hyperbolic ball of radius  $2 \operatorname{arcsinh}(e^{r_0}\sqrt{n-1}R/4)$ . Therefore,  $\bar{c}_2(s)$  is bounded above by

$$\left\lceil \frac{\operatorname{vol}(B(2 \operatorname{arcsinh}(e^{r_0}\sqrt{n-1}R/4) + 2 \operatorname{arcsinh}(e^{-(s-1)r_0}\sqrt{n-1}R/4) + \bar{\varepsilon}_0))}{\operatorname{vol}(B(i_M/2))} \right\rceil.$$

Since both,  $\bar{c}_1(s)$  and  $\bar{c}_2(s)$  are non-increasing in  $s$ , we conclude the claim by setting  $\bar{c} := \bar{c}_1(\bar{s}_0 + 1)\bar{c}_2(\bar{s}_0 + 1)$ .  $\square$

Analogously to the proof of Lemma 5.5, the previous Lemma yields the following.

**Lemma 5.13.** *Assume that condition (5.10) is satisfied. Then, for  $m \in \mathbb{N}$ ,*

$$\begin{aligned} |\mathcal{Q}^g(m+1)| &\geq (k - \mathbf{1}_{\{\bar{\ell}(\bar{s}_0+1)=0\}}\bar{c}\lfloor\bar{\varphi}(0)\rfloor)|\mathcal{Q}^g(m)| \\ &\quad - \bar{c} \cdot \sum_{j=\max(\bar{\ell}(\bar{s}_0+1),1)}^m (\lfloor\bar{\varphi}(j)\rfloor - \lfloor\bar{\varphi}(j-1)\rfloor)|\mathcal{Q}^g(m-j)|. \end{aligned}$$

*Proof.* Recall the definition of the set  $H_j = \{C_{is} \in \mathcal{C}_{m+1} : i+s-1 = m-j\}$  in (5.3). Since  $\bar{\ell}$  is non-decreasing we have  $j = m+1 - (i+s) = \bar{\ell}(s) \geq \bar{\ell}(\bar{s}_0+1)$  if  $s > \bar{s}_0$ .  $\square$

Finally, if moreover condition (4.4) is satisfied, then the same inductive proof as in Lemma 5.6 shows that the number of good cubes in  $\mathcal{Q}^g(m+1)$  increases in  $m+1$  by the factor  $c > 1$ ; see (5.4). Lemma 5.10.(2) then shows the existence of a  $\bar{\varphi}$ -aperiodic geodesic  $\bar{\gamma} : \mathbb{N} \rightarrow M$ . Thus, we have shown the following.

**Lemma 5.14.** *Assume that conditions (4.3) and (4.4) are satisfied. Then, for  $m \in \mathbb{N}$ ,  $|\mathcal{Q}^g(m)| \geq c^m$ . In particular, there exists a  $\bar{\varphi}$ -aperiodic geodesic  $\bar{\gamma} : \mathbb{N} \rightarrow M$  with parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ .*

Now, let  $\bar{\gamma} : \mathbb{N} \rightarrow M$  be a  $\bar{\varphi}$ -aperiodic geodesic (with parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ ) and let  $\gamma : \mathbb{R} \rightarrow M$  be the corresponding extended geodesic. Consider the sequence  $v^n := \phi^n \gamma'(r_0)$ ,  $n \in \mathbb{N}$ , in the compact space  $SM$  and let  $\gamma_0$  be an accumulation point. The space of unit speed geodesics (identified with  $SM$ ) is endowed with the topology of uniform convergence on bounded sets. Therefore note that a sequence  $v^n$  converges to  $v$  in  $SM$  if and only if for every  $l \geq 0$  and every  $\tau > 0$  there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $d(\gamma_{v^n}(t), \gamma_v(t)) < \tau$  for every  $t \in [-l, l]$ . Therefore  $\bar{\varphi}$ -aperiodicity can be shown to be a closed condition (similarly as in Lemma 2.4). Since  $\bar{\gamma}_{v^n}$  is  $\bar{\varphi}$ -aperiodic beginning at  $t_n \geq -(n-1)$  (with parameters  $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ ), it follows that  $\bar{\gamma}_0 : \mathbb{Z} \rightarrow M$  is  $\bar{\varphi}$ -aperiodic. This completes the proof of Theorem 4.6.

**5.3. Proof of Theorem 4.3.** For  $\delta \in (0, 1)$  choose  $\bar{\delta} \in [\delta, 1)$  such that for  $r_0 = \ln(3 - \bar{\delta})$  we have  $\ln(3 - \bar{\delta}) + \varepsilon_0 < i_M$ . Note that  $\tilde{\delta} = \bar{\delta} \ln(2)/\ln(3 - \bar{\delta}) \rightarrow 1$  as  $\bar{\delta} \rightarrow 1$  and assume therefore that  $\tilde{\delta} > \delta$ . For  $l \geq 0$  let  $\bar{\psi}(l) = 2^{\bar{\delta}(n-1)l}$  so that its right inverse  $\lceil \frac{1}{\bar{\delta}(n-1)\ln(2)} \ln(s) \rceil$  is an unbounded function. Then, for  $c = \frac{1}{2}(2^{n-1} + 2^{\bar{\delta}(n-1)})$ , we have that for sufficiently large  $\bar{s}_0 = \bar{s}_0(\bar{\delta}, n, i_M, \varepsilon_0) \in \mathbb{N}_0$  the conditions (4.3) and (4.4) are satisfied. Thus, from Theorem 4.6 there exists a discrete geodesic  $\bar{\gamma} : \mathbb{Z} \rightarrow M$  which is  $\bar{\psi}$ -aperiodic with respect to  $(\bar{s}_0, r_0 + \varepsilon_0, r_0)$ . From Lemma 4.5 we obtain that  $\gamma : \mathbb{R} \rightarrow M$  is continuously  $\psi$ -aperiodic with parameters  $s_0 = (\bar{s}_0 + 1)r_0$  and  $\varepsilon_0$ , where for  $l \geq r_0$ ,

$$\begin{aligned} \psi(l) &= \ln(3 - \bar{\delta}) \cdot \bar{\psi}\left(\frac{l}{\ln(3 - \bar{\delta})} - 1\right) - \ln(3 - \bar{\delta}) \\ &= \frac{\ln(3 - \bar{\delta})}{2^{\bar{\delta}(n-1)}} e^{\frac{\bar{\delta} \ln(2)}{\ln(3 - \bar{\delta})}(n-1)l} - \ln(3 - \bar{\delta}) \\ &= \left(\frac{\ln(3 - \bar{\delta})}{2^{\bar{\delta}(n-1)}} - \frac{\ln(3 - \bar{\delta})}{e^{\bar{\delta}(n-1)l}}\right) e^{\bar{\delta}(n-1)l} \\ &=: c(\tilde{\delta}, l) \cdot e^{\bar{\delta}(n-1)l} = c(\tilde{\delta}, l) \varphi_{\tilde{\delta}}(l). \end{aligned}$$

Note that  $c(\tilde{\delta}, l)$  is increasing in  $l$  and we restrict  $\psi$  to the interval  $[l_1, \infty)$  for some  $l_1 > \ln(3 - \tilde{\delta})$  such that  $c(\tilde{\delta}, l_1) > 0$ .

We now translate the minimal shift  $s_0$  into the minimal length  $l_0$ . Let to this end  $N := \lceil \frac{s_0}{2i_M} \rceil$ . Assume that for some  $t_0$  we have  $d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0$  for all  $0 \leq t \leq l$  where  $l \geq \max\{l_1, 3Ns_0 + 2i_M\} =: l_0$ .

First, we assume that  $s \leq s_0$ . Note that the function  $t \mapsto d(\gamma(t_0 + t), \gamma(t_0 + s + t))$  is not only convex but decreases and increases exponentially (see [3]) so that we have  $d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0/4$  for all  $s' \leq t \leq l - s'$  where  $s'$  is sufficiently large, say  $s' = 2i_M$ . The closing lemma implies the existence of a closed geodesic nearby; in fact, we will prove the following Lemma.

**Lemma 5.15.** *In this setting, there exists a closed geodesic  $\alpha$  of period  $p \leq s + \varepsilon_0/4$  such that (up to parametrization of  $\alpha$ ),*

$$d(\alpha(t), \gamma(t_0 + s' + t)) < \varepsilon_0/2 \quad \text{for all } 0 \leq t \leq s + l - 2s' - \varepsilon_0.$$

Let  $N' = \lceil s_0/p \rceil \in \mathbb{N}$  be the smallest integer such that  $N'p > \bar{s}_0$  and note that  $2Ns \geq N'p$ . We then have by the triangle inequality,

$$\begin{aligned} & d(\gamma(t_0 + s' + t), \gamma(t_0 + s' + N'p + t)) \\ & \leq d(\gamma(t_0 + s' + t), \alpha(t)) + d(\gamma(t_0 + s' + N'p + t), \alpha(t)) < \varepsilon_0 \end{aligned}$$

for all  $0 \leq t \leq l - 2s' - N'p + s$  and in particular for all  $0 \leq t \leq l - 2s' - 2Ns_0$ . Thus,

$$2Ns \geq N'p > c(\tilde{\delta}, l_1) \varphi_{\tilde{\delta}}(l - 2s' - 2Ns_0) = \frac{c(\tilde{\delta}, l_1)}{e^{\tilde{\delta}(n-1)(2s'+2Ns_0)}} \varphi_{\tilde{\delta}}(l),$$

and we can find a positive constant  $c_0 = c_0(\tilde{\delta}, i_M, n, \varepsilon_0)$  such that  $s > c_0 \varphi_{\tilde{\delta}}(l)$ .

In the case when  $s > s_0$ , we have

$$s > c(\tilde{\delta}, l_1) \varphi_{\tilde{\delta}}(l) \geq c_0 \varphi_{\tilde{\delta}}(l).$$

Finally, since  $\delta < \tilde{\delta}$ , we restrict if necessary to  $\tilde{l}_0 \geq l_0$  such that  $c_0 \varphi_{\tilde{\delta}}(l) \geq \varphi_{\delta}(l)$  for all  $l \geq \tilde{l}_0$ . The proof of Theorem 4.3 is finished by the proof of Lemma 5.15.

*Proof of Lemma 5.15.* We consider the setting of the proof of Theorem 4.6. Let now  $d_M$  be the distance function on  $M$  and recall that we have  $d_M(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0/4$  for all  $s' \leq t \leq l - s'$ , where  $s' = 2i_M > 2\ln(2)$ . We denote a lift of the segment  $\gamma$  on  $[t_0 + s', t_0 + l - s']$  by  $\beta$  and let the endpoints of  $\beta$  be  $x_1$  and  $x_2$ . Since  $\varepsilon_0 < i_M$ , there exists an isometry  $\psi \in \Gamma$  such that  $d(\beta, \psi(\beta(t))) < \varepsilon_0/4$  for all  $t \in [t_0 + s', t_0 + l - s']$  and in particular,  $d(x_i, \psi x_i) < \varepsilon_0/4$  for  $i = 1, 2$ . Let  $\tilde{\alpha}$  be the axis of  $\psi$  and denote by  $d_1 = d(\tilde{\alpha}, x_1)$  and  $d_2 = d(\tilde{\alpha}, x_2)$ . We first show that  $d_1$  is close to  $d_2$  in the following sense. Namely, the displacement function  $d_\psi(\cdot) = d(\psi \cdot, \cdot)$  grows at least linearly in the distance to  $\tilde{\alpha}$ . Since  $s - \varepsilon_0/4 \leq d_\psi(x_i) \leq s + \varepsilon_0/4$  for  $i = 1, 2$  we see that  $|d_1 - d_2|$  is bounded by a constant depending only on  $\psi$ ,  $s$  and  $\varepsilon_0$ .

Now, if we show that  $d_i < \varepsilon_0/2$  for  $i = 1, 2$ , then the proof follows by convexity of the distance function. We show this for  $d_1$ . Since  $d_1$  is close to  $d_2$  and  $l$  is large, the distance function  $t \mapsto d(\beta(t), \tilde{\alpha}(t))$  decreases exponentially on  $[0, s']$ , where  $\tilde{\alpha}$  is parametrized such that  $\tilde{\alpha}(0)$  equals the orthogonal projection  $\bar{x}_1$  of  $x_1$  on the convex set  $\tilde{\alpha}$ . Moreover,  $s'$  is large and thus  $d(\tilde{\alpha}, \beta(s')) < d_1/2$ . The orthogonal projection of  $\psi(x_1)$  on  $\tilde{\alpha}$  is given by  $\psi(\bar{x}_1)$ . Hence,  $d(\psi(x_1), \tilde{\alpha}(s')) \geq d(\psi(x_1), \psi(\bar{x}_1)) = d_1$ . On the other hand, we have by the triangle inequality  $d(\psi(x_1), \tilde{\alpha}(s')) \leq d(\psi(x_1), \beta(s')) + d(\beta(s'), \tilde{\alpha}(s')) < d_1/2 + \varepsilon_0/4$ . Thus,  $d_1 < d_1/2 + \varepsilon_0/4$  and the claim follows.  $\square$

## REFERENCES

- [1] V.I. Bernik and M.M. Dodson. *Metric Diophantine Approximation on Manifolds*, volume 137. Cambridge University Press, 1999.
- [2] M. D. Boshernitzan. Quantitative recurrence results. *Inventiones mathematicae*, 112:617–631, 1993.
- [3] M. R. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature*. Springer, 1999.
- [4] P. Eberlein. *Geometry of Nonpositively Curved Manifolds*. University Of Chicago Press, 1994.
- [5] M. Einsiedler and T. Ward. *Ergodic Theory, with a view towards Number Theory*. Springer, 2011.
- [6] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory*.
- [7] A. Haas. Geodesic cusp excursions and metric diophantine approximation. *Math. Res. Let.*, 16:67–85, 2009.
- [8] E. Heintze and H.C. Im Hof. Geometry of horospheres. *Journal of Differential Geometry*, Volume 12(Number 4):481–491, 1977.
- [9] S. Hersonsky and F. Paulin. Diophantine approximation for negatively curved manifolds. *Mathematische Zeitschrift*, 241:181–226, 2002.
- [10] S. Hersonsky and F. Paulin. On the almost sure spiraling of geodesics in negatively curved manifolds. 85:271–314, 2010.
- [11] M. Morse and G. Hedlund. Unending chess, symbolic dynamics and a problem in semigroups. *Duke Math. Journal*, 11:1–7, 1944.
- [12] D. Ornstein and B. Weiss. Entropy and recurrence rates for stationary random fields. *IEEE Transactions on Information Theory*, 48(6), 2002.
- [13] J. Parkkonen and F. Paulin. Prescribing the behaviour of geodesics in negative curvature. *Geometry and Topology*, 14:277–392, 2010.
- [14] J. Parkkonen and F. Paulin. Spiraling spectra of geodesic lines in negatively curved manifolds. *Mathematische Zeitschrift*, 268:101–142, 2011.
- [15] S.J. Patterson. Diophantine approximation in fuchsian groups. *Philos. Trans. Roy. Soc. London. Series A*, 282:527–563, 1976.
- [16] J. C. Robinson. *Dimensions, Embeddings and Attractors*, volume 186. Cambridge University Press, 2011.
- [17] D. Sullivan. Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. *Acta Math.*, 149:215–237, 1982.
- [18] S.J. Velani. Diophantine approximation and hausdorff-dimension in fuchsian groups. *Math. Proc. Camb. Phil. Soc.*, 113:343–354, 1993.
- [19] M. Gromov W. Ballmann and V. Schroeder. *Manifolds of nonpositive curvature*, volume 61. Birkhäuser, 1985.
- [20] P. Walters. *Introduction to Ergodic Theory*, volume 79. Springer, 1981.

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